An Invitation to Graphical Tensor Methods

Exercises in Graphical Vector and Tensor Calculus and More

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This is the supplementary material for "Boosting Vector Calculus with the Graphical Notation." It is written for students and introduces instructive and practical examples regarding graphical vector and tensor algebra and calculus in the physics context. Notable topics, including the ones mentioned in the main article, are as follows. Graphical proof of the Jacobi identity is discussed in **P6**a and its sequels in the following two sections. Next, most importantly, graphical proof of first and second derivative vector calculus identities are covered in Sections I B 1 and I B 2. Then, the strength of the graphical technique of tensor calculus is elaborated throughout Section I C. Graphical notation for general multi-index quantities is introduced in Section I C 2. Do not miss the next section, Section I C 3, to learn the "arrow pushing" that enables to efficiently denote and explain the invariance property of tensorial expressions. Finding the oblivious $\delta^{(3)}(\vec{r})$ term and the calculus of multipolar fields are discussed throughout Sections I C 5 and I C 6. Section I A 4 introduces permutation symmetry of indices. Refer to the comments in **A67**b for an instructive example of diagrammatic perturbation in statistical mechanics. Lastly, Section I D invites the reader into the world of Feynman diagrams.

Difficulty Levels
Without mark: essential or elementary.
With [⋆] : intermediate level, adequate for advanced mathematical physics courses.
With [!] : challenge only if you find yourself relishing graphical reasoning or you want to be a true "virtuoso."

I. Problems

A. Graphical Vector Algebra

1. Basic Translation Tasks

A possible calculation strategy is to translate the plaintext equations into graphical notation, proceed by graphical manipulations, then return to the plaintext notation if the answer is required to be written in the plaintext notation. If you become a true "bilingual," translating one to the other will not bother you anymore. The problems in this section is for practicing such translation tasks.

P1 Draw corresponding diagrams for each of following plaintext expressions, and then give at least one alternative reading in index-free plaintext notation.

(a) $\vec{A}\cdot(\vec{B}\times\vec{C})$ (b) $\vec{A}\times(\vec{B}\times\vec{C})$
(c) $(\vec{A}\times(\vec{B}\times\vec{C}))\cdot\vec{D}$

P2 Draw corresponding diagrams for each of following plaintext expressions, and then give at least one alternative reading in the index notation.

(a) $A_iB_lC_m\epsilon_{ijk}\epsilon_{klm}$ (b) $A_mB_k\delta_{ij}\delta_{ml}\epsilon_{lkj}$
(c) $A_iB_jC_k\epsilon_{imj}\epsilon_{lkm}$ (d) $A_lB_m\epsilon_{ilj}\epsilon_{jmi}$
(e) $A_iB_kC_n\epsilon_{ijm}\epsilon_{kln}\epsilon_{jlm}$

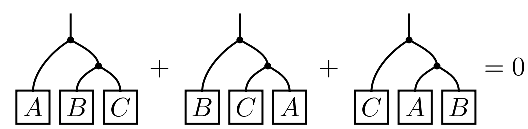
P3 Translate diagrams in Table I into plaintext (index-free or index) notation. Attach index markers to the diagrams adequately if needed.

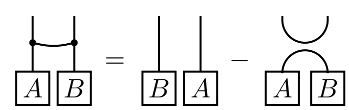
P4 Use *the economy of the graphical notation* to show that $(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = \vec{A}\cdot(\vec{B}\times(\vec{C}\times\vec{D}))$: that is, show that the both sides of the equation are just two different readings of an identical diagram.

P5 Graphically represent the following plaintext equations.

(a) $(\vec{A}\times\vec{B})\times(\vec{C}\times\vec{D}) = (\vec{D}\times\vec{C})\times(\vec{A}\times\vec{B})$
(b) $\epsilon_{ikj}\epsilon_{jli}A_kB_l = -\epsilon_{ijl}\epsilon_{ijk}A_lB_k$
(c) $\epsilon_{ijk}A_iB_jC_k = \epsilon_{ijk}A_jB_kC_i$

P6 Translate the following graphical equations into the plaintext notation. (You don't need to prove them.)

(a) 

(b) 

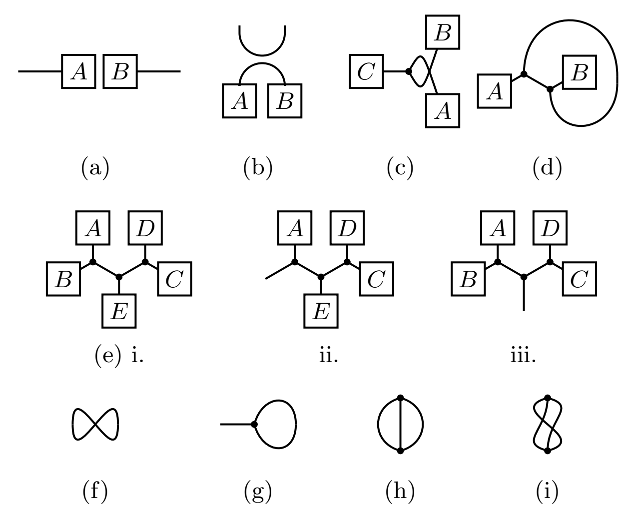
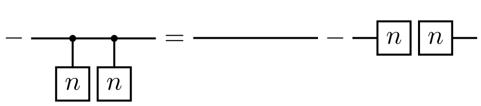


TABLE I. Diagrams for **P3**.

P7 Prove that $\vec{A}\times\vec{A} = 0$ in the graphical notation, and then in the plaintext notation. Show the correspondence between two proofs by pairing each step.

P8 Prove the following graphical identity for a unit vector $\vec{n}$ (i.e., $\vec{n}\cdot\vec{n} = 1$) in the graphical notation, then in the plaintext notation. Show the correspondence between two proofs by pairing each step. [*Note*: *this is a linear map that projects out the component parallel to* $\vec{n}$.]

 (1)

P9 Confirm that the above $\vec{n}$-projector is idempotent by diagrams. That is, if we call Eq. (1) by " ," show that = .

2. See the Bones, Attach the Flesh Pieces

P10 Show that $(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})$ and the BAC-CAB rule $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$ are daughters of the same tensorial structure, . List all the algebraic identities that originate from , in the plaintext notation.

P11 **P6**a is called the "Jacobi identity."

(a) Prove the Jacobi identity using . Reduce each term into smaller units, and show that they cancel out one another.

(b) Generate all vector algebraic identities that are identical to the Jacobi identity. [*Hint*: *extract the bones first, then attach the flesh pieces one by one.*]

3. Epsilon Networks (Essential)

Now, we abstract out the flesh pieces and investigate the world of bones. First, consider identities involving two cross product machines.

P12 By joining (contracting) the terminals of , we can obtain derived identities.

(a) Join two terminals of , and obtain a nontrivial identity. [*Answer*: $= -2$.]

(b) Join the two terminals of the identity you obtained from the previous question. What identity do you get? [*Answer*: $= -6$.]

(c) Prove the identities obtained in **P12**a and **P12**b using the plaintext notation.

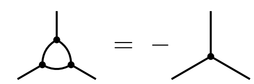
Next, let's move on to identities involving three cross product machines.

P13 Have some fun playing with the cross product machines. Take out three cross product machines from the toy box and assemble them into various configurations. Find connected nonzero expressions as many as you can. For example, exclude diagrams such as or , which are disconnected or zero due to the knot at its rightmost part, respectively.

P14 How can you classify the diagrams you obtained in **P13**? A possible criterione is to divide them in terms of the number of loops they contain. If a diagram doesn't have any loops, it is a "tree-level diagram." Then we have "one-loop diagrams," "two-loop diagrams," and so on. Classify your diagrams by their loop numbers.

P15 According to the terminology for Feynman diagrams, "one-particle irreducible diagrams" (abbreviated as "1PI diagrams") are diagrams that cannot be split into two pieces by cutting an internal line. For example, is not a 1PI diagram (or), but is. Find all 1PI diagrams that can be built from three cross product machines and reduce them into lower-epsilon terms until it becomes impossible to break them down further by the two-epsilon identities in **P12** (because the two-epsilon identities reduces a two-epsilon subdiagram into a zero-epsilon subdiagram, there will remain one epsilon in each term of the final expression).

How do three-epsilon networks play a role in real use? First, consider the following three-terminal one-loop diagram, which you may have obtained in **P15**.

 (2)

P16 The proof of the Jacobi identity in **P11**a proceeds with breaking down all the terms into expressions of lower-epsilon level (expressions that has less cross product machines). The resulting six terms cancels one another. Instead of that, however, start from and rather "climb up the stairs" so that move on to four-epsilon expression, then to the desired result. Use Eq. (2) and a two-epsilon identity during the process. Keep in mind the lesson here: sometimes, going up can provide a shortcut.

Meanwhile, a tree-level three-epsilon diagram also comes into play in vector algebra.

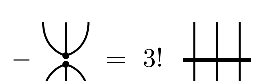
P17 Construct a vector algebra identity based on the two possible expansion of a three-epsilon composition . First, apply to the adjacent cross product machines on the left side, then on the right side. [*Answer*: $(\vec{A}\cdot\vec{B}\times\vec{C})\vec{D} = \vec{A}(\vec{B}\cdot\vec{C}\times\vec{D}) - \vec{B}(\vec{A}\cdot\vec{C}\times\vec{D}) + \vec{C}(\vec{A}\cdot\vec{B}\times\vec{D})$.]

If you want, you can spend your time playing with higher-epsilon diagrams. Penrose[1], for example, provides more identities, so you may want to check it.

4. Epsilon Networks (Advanced)

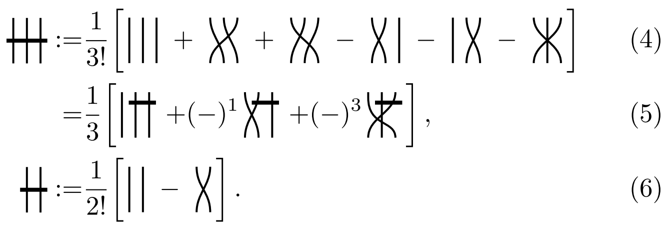
This section introduces the concept of the antisymmetrizer and the consequences of permutation symmetry of diagrams (Jucys-Levinson-Vanagas theorem or Schur's lemma) in a concise manner. There are good references[2,3] regarding this subject, so we rather mention only the essentials.

Until now, has been utilized as the fundamental identity that enables us to proceed all calculations. It was instructive to introduce the graphical vector algebra by such manner, instead of giving all the details in a rigorous and logically preferred order from the beginning. In fact, however, the fundamental two-epsilon identity is not but the following,

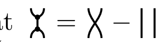


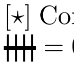
$$= 3! \quad ,$$ (3)

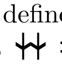
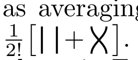
where a thick line over three Kronecker delta lines means that they are totally antisymmetrized, i.e.



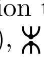
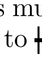
The minus sign in Eq. (3), the "master" identity, is due to our convention that reads terminals of a cross product machines anti-clockwise. 3! is due to normalization convention, which is optional. In the plaintext notation, Eq. (3) reads "$\epsilon^{ijk}\epsilon_{lmn} = 3!\delta^i_{[l}\delta^j_m\delta^k_{n]}$."[4] Antisymmetrizing four Kronecker deltas gives zero in three-dimensions, i.e.  $= 0$, because there cannot exist four linearly independent vectors.

P18 [$\star$] Confirm that  can be derived from Eq. (3) by diagrams. (How is it compared to using the plaintext notation?)

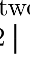
P19 [$\star$] Confirm that the identity in **P17** can be derived from  $= 0$.

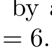
For later purposes, we define the symmetrizer too. It is defined as averaging all permutations of indices. For example,  := .  is defined similarly: just change all minuses into pluses in Eq. (4).

P20 Confirm that .

One can prove Eq. (3) from the identity in **P17**,[2] but the calculation is tedious. Instead of that, the observation that the three upper terminals of the left hand side of Eq. (3), , are totally antisymmetric as much as the three lower terminals implies that it is proportional to .[5] Then, the proportionality constant can be determined by appropriate contraction of terminals.

P21 [$\star$] Prove Eq. (3) by such "appealing to symmetry" method we just described.

P22 [$\star$] This strategy is readily applied to other multi-terminal cases. Schur's lemma says that any two-terminal epsilon network is proportional to Kronecker delta. Prove  $= -2$  from Schur's lemma.

P23 [$\star$] Prove Eq. (2) by appealing to permutation symmetry, provided that  $= 6$.

P24 [$\star$] Any one-terminal epsilon network (which is called a "tadpole diagram" in the terminology of Feynman diagrams) must be equal to zero. Explain why.

Note that students themselves, by doodling the graphical algebra, can find the fact that "a compound n-terminal object that has a permutation symmetry can be reduced into a simpler expression of the same symmetry up to a proportionality constant", as mentioned in the main article.

B. Graphical Vector Calculus

1. First Derivatives

P25 There are six first derivative identities, and the missing ones in the main article are the following two. Prove these by the graphical notation.

(a) $\nabla\cdot\big(f\vec{A}\big) = f\,\nabla\cdot\vec{A} + \nabla f\cdot\vec{A}$

(b) $\nabla\times\big(f\vec{A}\big) = f\,\nabla\times\vec{A} + \nabla f\times\vec{A}$

P26 Express the following identity by diagrams.

$$(\nabla\times\vec{A})\cdot(\nabla\times\vec{B}) = \frac{\partial A_j}{\partial x_i}\frac{\partial B_j}{\partial x_i} - \frac{\partial A_j}{\partial x_i}\frac{\partial B_i}{\partial x_j} \tag{7}$$

P27 Prove the following identities by diagrams. The expression $\dot{\nabla}\big(\dot{\vec{A}}\cdots\big)$ means that the differentiation operates on $\vec{A}$ only (Hestenes' overdot notation).[6]

(a) $\nabla\big(\vec{A}\cdot(\vec{B}\times\vec{C})\big) = \dot{\nabla}\big(\dot{\vec{A}}\cdot(\vec{B}\times\vec{C})\big) + \dot{\nabla}\big(\vec{A}\cdot(\dot{\vec{B}}\times\vec{C})\big) + \dot{\nabla}\big(\vec{A}\cdot(\vec{B}\times\dot{\vec{C}})\big)$

(b) $\dot{\nabla}\big(\dot{\vec{A}}\cdot(\vec{B}\times\vec{C})\big) = \big((\vec{B}\times\vec{C})\cdot\nabla\big)\vec{A} + (\vec{B}\times\vec{C})\times(\nabla\times\vec{A})$

P28 Calculate $\nabla(\vec{n})$, where $\vec{n} := \vec{r}/r$ is the unit radial vector. [*Answer*: r^{-1} *times the* $\vec{n}$*-projector, Eq.* (1)] Can you give this result a geometric interpretation?

P29 Graphically represent the identity "$\nabla\cdot(\vec{n}/r^2) = 4\pi\delta^{(3)}(\vec{r})$."

2. Second Derivatives

We already mentioned in the main article that $(\nabla\times\nabla)$ vanishes as an operator identity for well-behaved tensor fields, i.e., $(\nabla\times\nabla)f = 0$, $(\nabla\times\nabla)\vec{A} = 0$, or $(\nabla\times\nabla)$(any well-behaved tensor field) $= 0$.

P30 The following three are second derivative identities that are commonly mentioned in the literatures. Prove these by the graphical notation.

(a) $\nabla\times\nabla f = 0$, $\nabla\cdot(\nabla\times\vec{A}) = 0$

(b) $\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$

P31 We are now going to study the Laplacian operator. Prove the following identities by diagrams.

(a) $\nabla^2(f\,g) \;=\; g\,\nabla^2 f + 2\nabla f\cdot\nabla g + f\,\nabla^2 g$

(b) $\nabla^2(f\vec{A}) = \vec{A}\,\nabla^2 f + f\,\nabla^2\vec{A} + 2(\nabla f\cdot\nabla)\vec{A}$

(c) $\nabla^2(\vec{A}\cdot\vec{B}) = \vec{A}\cdot\nabla^2\vec{B} - \vec{B}\cdot\nabla^2\vec{A} + 2\nabla\cdot\big((\vec{B}\cdot\nabla)\vec{A} + \vec{B}\times(\nabla\times\vec{A})\big)$

P32 [$\star$] Calculate the following expressions by diagrams. Note that $\nabla^2\vec{r} = 0$.

(a) $\nabla^2 r$, $\nabla^2\vec{n}$ (b) $\nabla^2 z$

(c) $\nabla^2(r^2)$, $\nabla^2(3z^2 - r^2)$, $\nabla^2(x^2 - y^2)$, $\nabla^2(xy)$

(d) $\nabla^2\big(z(5z^2 - 3r^2)\big)$, $\nabla^2\big(x(5z^2 - r^2)\big)$, $\nabla^2\big(z(x^2 - y^2)\big)$, $\nabla^2\big(x(x^2 - 3y^2)\big)$

P33 Proving Bernoulli equation for the case of incompressible potential flow involves the following index gymnastics: $(\vec{u}\cdot\nabla)\vec{u} = \frac{1}{2}\nabla\big(\vec{u}\cdot\vec{u}\big)$, where $\vec{u}(\vec{r})$ is a vector field that can be written as $\vec{u}(\vec{r}) = \nabla\Phi(\vec{r})$ for some scalar field $\Phi(\vec{r})$. Prove this by diagrams.

P34 When deriving the vorticity equation, one takes curl to the Navier-Stokes equation. Prove that for a vector field $\vec{u}(\vec{r})$, $\nabla\times\big((\vec{u}\cdot\nabla)\vec{u}\big) = (\vec{u}\cdot\nabla)\big(\nabla\times\vec{u}\big) + (\nabla\cdot\vec{u})(\nabla\times\vec{u}) - \big((\nabla\times\vec{u})\cdot\nabla\big)\vec{u}$ using diagrams.

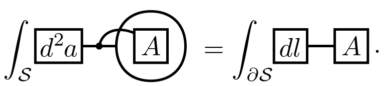
Identities of higher order derivatives can be obtained by combinations of first and second order identities. Since they do not illustrate the unique power of graphical calculus much, we choose not to discuss higher-order identities in detail.

P35 Graphically represent the operator identity $\nabla^2 \nabla = \nabla \nabla^2$ for well-behaved tensor fields.

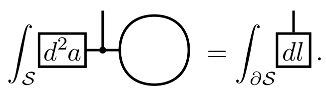
P36 When studying spherical waves, a useful fact is that if $\psi(\vec{r})$ satisfies the Helmholtz equation, $(\nabla^2 + k^2)\psi(\vec{r}) = 0$, then $\vec{r} \times \nabla\psi(\vec{r})$ satisfies the vector Helmholtz equation, $(\nabla^2 + k^2)\big(\vec{r} \times \nabla\psi(\vec{r})\big) = 0$. Prove this using the graphical notation.

3. *Graphical Notation for Integral Calculus*

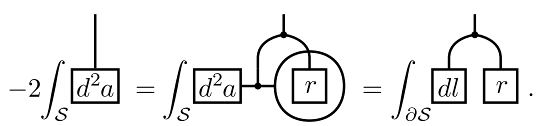
According to the graphical notation for scalars, vectors, and ∇ discussed until now, the curl theorem, $\int_{\mathcal{S}} d^2\vec{a} \cdot \nabla \times A = \int_{\partial\mathcal{S}} d\vec{l} \cdot \vec{A}$, can be written as the following.

 (8)

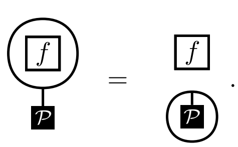
∂ is the boundary operator, and $\mathcal{S}$ is a surface. Here, one can observe that —[A] is rather a "dummy" vector field and extract the "essence" of the curl theorem as

 (9)

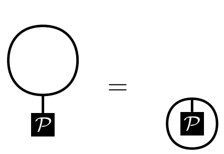
In the plaintext notation, Eq. (9) will be written as $\int_{\mathcal{S}} d^2\vec{a} \times \nabla[\cdots] = \int_{\partial\mathcal{S}} d\vec{l}[\cdots]$. Understanding the curl theorem in this form is in many ways useful. For example, inserting a scalar field $f(\vec{r})$ gives $\int_{\mathcal{S}} d^2\vec{a} \times \nabla f = \int_{\partial\mathcal{S}} d\vec{l} f$. Or, inserting $\vec{r}$ then taking cross product derives the vector area formula $\int_{\mathcal{S}} d^2\vec{a} = \frac{1}{2}\int_{\partial\mathcal{S}} \vec{r} \times d\vec{l}$ as

 (10)

Now, although it is not necessary, one would feel an "impulse" to express Stokes' theorems completely in diagrammatic terms. Let us see if we can, anticipating the insight that the graphical language will provide. One possible way of graphically denoting integrals is to treat dummy variables of integrals analogous to dummy indices in tensor expressions; the resulting notation shares the same ground with the graphical version of bra-ket notation.[7] However, when it comes to Stokes' theorems, we would like to suggest a different approach highlighting the trading between ∇ and ∂. In this approach, the gradient theorem, $\int_{\mathcal{P}} d\vec{l} \cdot \nabla f = \int_{\partial\mathcal{P}} f$, where $\partial\mathcal{P} = +$(endpoint of $\mathcal{P}$) $-$ (starting point of $\mathcal{P}$) for a path $\mathcal{P}$, is expressed as

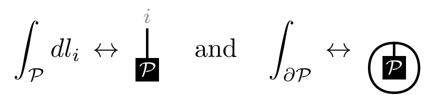
 (11)

Note that the scalar field f in Eq. (11) is also a "dummy" field as $\vec{A}$ in Eq. (8) did. Omitting [f], Eq. (11) becomes

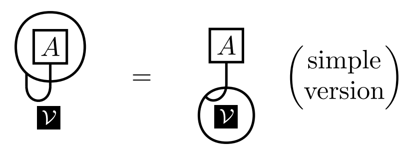
 (12)

i.e., flipping the balloon inside out. When a balloon swallows a manifold $\mathcal{P}$, it becomes "differentiated:" $\partial\mathcal{P}$. This is a graphical manifestation of the pairing of differential forms and chains in a de Rham-theoretic manner, namely, $\langle \mathcal{P}, df \rangle = \langle \partial\mathcal{P}, f \rangle$. When a manifold and a tensor field are met, they produce the "contraction," i.e., the value of integration of the tensor field on the manifold. Contraction between a manifold and a field is denoted just by juxtaposing them, where Stokes theorem translates to flipping the differentiation balloon inside out.

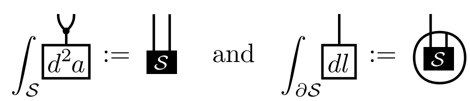
What about curl and divergence theorems? To be specific, Eq. (11) used the following substitution.

 (13)

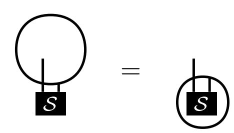
For the divergence theorem, $\int_{\mathcal{V}} d^3x\, \nabla \cdot \vec{A} = \int_{\partial\mathcal{V}} d^2\vec{a} \cdot \vec{A}$ for a volume $\mathcal{V}$, one can guess that a substitution $\int_{\mathcal{V}} d^3x \leftrightarrow$ [$\mathcal{V}$] can be made. Then, the left hand side of the divergence theorem can be written in a completely graphical form. After that, flipping the balloon will generate the right hand side. The result is the following.

 (14)

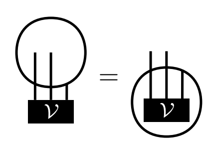
Then, it follows that $\int_{\partial\mathcal{V}} d^2a_i \leftrightarrow$ i—($\mathcal{V}$). In case of the curl theorem, substitutions

 (15)

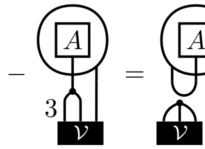
make Eq. (9) to be represented in the "flipping balloon" form.

 (16)

The origin of these substitution rules can be consistently explained by employing the framework of differential forms; this will interest more mathematically sophisticated readers. In fact, index gymnastics with [$\mathcal{P}$], [$\mathcal{S}$], and so on corresponds to manipulating differential forms. Our convention is to represent the integration over an n-dimensional manifold as a totally antisymmetric n-legged black rectangle. However, the divergence theorem Eq. (14) appears to be an exception, because the integration over a volume $\mathcal{V}$ was earlier denoted as a zero-legged object as $\int_{\mathcal{V}} d^3x \leftrightarrow$ [$\mathcal{V}$]. To be consistent with Eq. (12) and Eq. (16), the divergence theorem should be written as the following.

 (17)

[$\mathcal{V}$] is a totally antisymmetric three-legged object that denotes the integration over $\mathcal{V}$. To be concrete, its components may be identified with $3!\int d\lambda_1 d\lambda_2 d\lambda_3\, (\partial X_{[i}/\partial\lambda_1)(\partial X_j/\partial\lambda_2)(\partial X_{k]}/\partial\lambda_3)$, where $\vec{r} = \vec{X}(\lambda_1, \lambda_2, \lambda_3)$ is a parametrization of $\mathcal{V}$. Then, when an identification [$\mathcal{V}$] $= \frac{1}{3!}$ [$\mathcal{V}$] is made, index gymnastics show that Eq. (17), when contracted with , is equivalent to the form first introduced, Eq. (14) (exercise!). One of the equations that appears during the index gymnastics is the following.



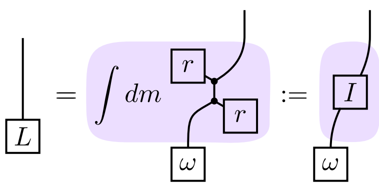
(18)

This corresponds to $\mathrm{d}(\frac{1}{2!}\epsilon_{ijk}A_k\,\mathrm{d}x_i \wedge \mathrm{d}x_j) = (\nabla\cdot\vec{A})\,\frac{1}{3!}\epsilon_{ijk}\mathrm{d}x_i \wedge \mathrm{d}x_j \wedge \mathrm{d}x_k$ in the calculus of differential forms, where d is the exterior derivative.

C. Graphical Tensor Calculus

1. Symmetric Rank-2 Tensors

Welcome to the world of tensors, finally. Graphically, tensors are nothing but multi-terminal objects: monopods, dipods, tripods, tetrapods, and so on. We already mentioned the graphical notation for symmetric dipods in the main article. Let us do a little stretching first with the inertia tensor. The inertia tensor of a rigid body is defined as follows.



(19)

The integral is over infinitesimal mass elements of the body, which is labelled by $\vec{r}$.

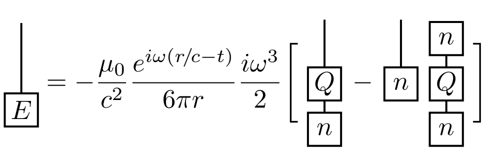
P37 Confirm that the inertia tensor is symmetric under swap-then-yanking of the external arms of the shaded expression in the middle of Eq. (19).

P38 The mass quadrupole moment Q_{ij}, a rank-2 symmetric tensor, is defined as $Q_{ij} = \int dm\, \frac{1}{2}(3x_ix_j - r^2\delta_{ij})$. Graphically represent this equation.

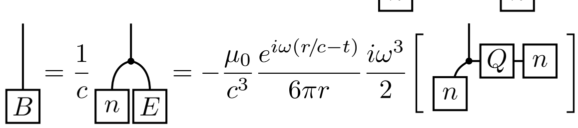
P39 Calculate the trace of the inertia tensor and the mass quadrupole moment, respectively, by diagrams. That is, calculate I_{ii} and Q_{ii}.

P40 Prove that $Q_{ij} = -\frac{3}{2}I_{ij} + \frac{1}{2}\delta_{ij}I_{kk}$ by diagrams.

Now, we move on to electromagnetism. Consider the electric quadrupole radiation of a monochromatic source $\rho(t,\vec{r}) = \varrho(\vec{r})e^{-i\omega t}$ and $\vec{J}(t,\vec{r}) = j(\vec{r})e^{-i\omega t}$ localized in $\mathcal{V}$ near the origin. Define the electric quadrupole moment with $\varrho(\vec{r})$, i.e., $Q_{ij} := \int_{\mathcal{V}} d^3x\, \varrho(\vec{r})\,\frac{1}{2}(3x_ix_j - r^2\delta_{ij})$. An examination of the orders of r and ω reveals that the radiation electromagnetic fields are given as follows.



(20)



(21)

P41 [⋆] Calculate the time-averaged Poynting vector, $\vec{S} = \frac{1}{2\mu_0}\mathrm{Re}\,\vec{E}^*\times\vec{B}$, with Eq. (20) and Eq. (21). This will yield a huge tensorial expression, which, however, can be reduced wisely with the guidance of the graphical notation. Decide which "knot" is the best to cut off first.

Before closing this section, we present another electromagnetism example that involves a considerable use of index gymnastics, while the graphical notation effectively boosts the speed and helps us grasp the contraction structure in a bird's eye view (cf. the process with the plaintext notation, given in the literature[8]).

P42 [!] Consider electric and magnetic fields $\vec{E}(t,\vec{r})$ and $\vec{B}(t,\vec{r})$ in a homogeneous but anisotropic linear medium that has a constant permittivity and permeability matrix of components ε_{ij} and μ_{ij} in the lab frame, respectively. The auxiliary fields are given by $D_i(t,\vec{r}) = \varepsilon_{ij}E_j(t,\vec{r})$ and $H_i(t,\vec{r}) = (\mu^{-1})_{ij}B_j(t,\vec{r})$, where $(\mu^{-1})_{ij}$ is the component of the inverse of the permeability matrix. Both the permittivity and the permeability matrices are symmetric.

(a) Graphically write Maxwell's equations in media without any free charges and currents. Introducing a new graphical notation for $\frac{\partial}{\partial t}$ depends on your choice.

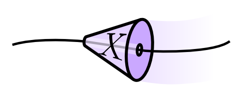
(b) Consider a plane wave solution and use the substitution $\frac{\partial}{\partial t} \to -i\omega$ and $\nabla \to i\vec{k}$. Let the amplitude of the electric field be $\vec{E}_0$. Let $v = k/\omega$ be the phase velocity. Find the equation that can be used to calculate v^2 when $\vec{E}_0$ and the direction of propagation, $\vec{u} := \vec{k}/k$, are known.

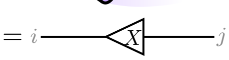
(c) Show that the direction of $\vec{E}_0$, can be found by the eigenvalue problem $W_{ij}(E_0)_j + v^2(E_0)_i = 0$, when $\vec{u}$ is known. Find the matrix W_{ij}.

(d) If you insist on working further without component-wise unpacking of the matrices, you will have to calculate W_{ii}, $\frac{1}{2!}\epsilon_{ijk}\epsilon_{imn}W_{jm}W_{kn}$, and the determinant $\frac{1}{3!}\epsilon_{ijk}\epsilon_{lmn}W_{il}W_{jm}W_{kn}$ in terms of $\vec{u}$ and the permittivity and permeability matrices to obtain an analytic formula of v^2. If you have some time to spare and have finished Section I C 4, have fun with calculating them.

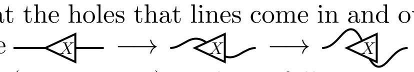
2. Graphical Notation for General Multi-index Quantities

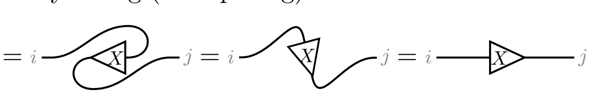
In the main article, we only introduced the graphical notation for rank-2 symmetric tensors. Now consider a general two-index quantity (need not be a tensor), X_{ij}. Suppose that X_{ij} has no symmetry so that X_{ji} is unrelated to X_{ij}. Then, following the philosophy of *self-explanatory design*, its graphical counterpart should be non-symmetrical in its two terminals, which becomes different when it is "swap-then-yanked."



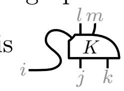
$X_{ij} =$ 

(22)

A designer's choice that suffices these requirements is a bead that has a non-symmetrical shape and slides along a line. The bead is rigid so that the holes that lines come in and out cannot be repositioned like . Instead, swap-then-yanking (transposing) works as follows.

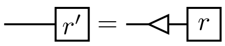
$X_{ji} =$ 

(23)

If $X_{ij} = X_{ji}$, the designer would advise us to use a symmetric shape for X, such as an oval, a rectangle, or a diamond. We used a square in the previous sections. Diagrams for general n-index quantities are the same. For instance, a possible graphical representation of a general 5-index quantity, K_{ijklm}, is .

3. Rotational Symmetry and Contra/Covariance

To be called a "tensor," a multi-index quantity must be endowed with invariance property. Suppose we actively transform the vector $\vec{r} = x_i \vec{e}_i$ by $\vec{r}\,' = R_{ij} x_j \vec{e}_i$, where (R_{ij}) is a special orthogonal matrix (i.e., encodes proper rotation) so that $(R_{ij})^\top = (R_{ij})^{-1}$ and $\det(R_{ij}) = +1$. In the graphical notation,

 (24)

where R_{ij} := instead of (triangle labelled R) to avoid clutter. (Let blank triangles always denote the rotation matrix from now on.)

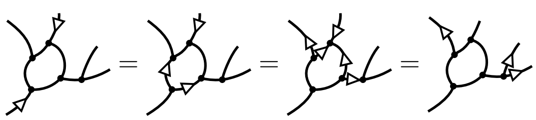
P43 R_{ij} satisfies $R_{ik}R_{jk} = \delta_{ij}$ and $R_{ki}R_{kj} = \delta_{ij}$. Graphically represent these equations. [*Answer*: diagrams. *Opposite arrowheads are pair created or annihilated!*]

P44 There is one more constraint: $\det([R_{ij}]) = +1$.

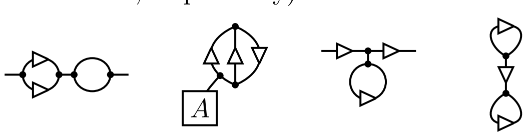
(a) Show that the diagram is totally antisymmetric in its three terminals so that is equal to the diagram.

(b) Using $\det([R_{ij}]) = \frac{1}{3!}\epsilon_{ijk}\epsilon_{lmn}R_{il}R_{jm}R_{kn}$ and Eq. (3), conclude that the diagrams are equal.

(c) Confirm the following "propagation of arrowheads" or "arrow pushing" holds.

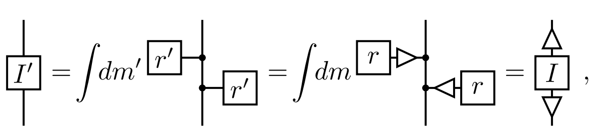


P45 [!] This problem is just for fun. Simplify the following diagrams (minimize the number of cross product machines and arrowheads, respectively).



(a) (b) (c) (d)

When $\boxed{r} \to$ (arrowhead) $\boxed{r}$, a tensor dresses outgoing arrowheads in its all terminals. For example, the inertia tensor defined by Eq. (19) transforms by the following, so it is indeed a tensor.

 (25)

P46 Confirm the last equality in Eq. (25).

P47 Prove that the mass quadrupole moment defined in **P38** is a tensor.

P48 When $\vec{A}$ and $\vec{B}$ are vectors, i.e., $A'_i = R_{ij}A_j$ and $B'_i = R_{ij}B_j$, show that $\vec{A}\cdot\vec{B}$ and $\vec{A}\times\vec{B}$ are a scalar and a vector, respectively, using the graphical notation.

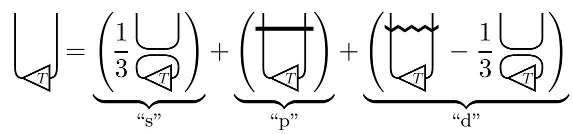
P49 [$\star$] Consider a rank-2 tensor T_{ij}.

(a) Show that $\operatorname{tr} T = T_{ii}$ is a scalar.

(b) Show that $T_{[ij]} = \frac{1}{2!}(T_{ij} - T_{ji})$ is a tensor and in fact has three independent components so that it can be encoded into a form of a vector (yet it is not exactly a vector). [*Hint*: *Do some combinatorics. Then consider* $\epsilon_{ijk}T_{jk}$. *Is it a vector?*]

(c) For $T_{ij} = A_i B_j$, calculate $\operatorname{tr} T$ and the vector you got in the last question. What are they?

P50 [$\star$] *Decomposition of a rank-2 tensor into irreducible representations of* $\mathsf{SO}(3)$. As you might already have caught the idea in the previous problem, a rank-2 tensor T_{ij} has many faces: the scalar times δ_{ij} (trace-only part), the part that can be encoded as a vector (antisymmetric part), and lastly, the remainder, a symmetric traceless rank-2 tensor. Thus, we have *decomposed* T_{ij} *into parts that transforms differently under rotation.* Show that this can be written as Eq. (26). Check that the numbers of independent components are 1, 3, and 5 for the three parts, respectively. These numbers may remind you of the degeneracies of spin-0, 1, and 2 angular momentum eigenstates. In fact, we call the three parts "spin-0," "spin-1," and "spin-2" part, respectively.



(26)

If you want to distinguish between contravariant and covariant indices, restrict the terminals to be always vertical. Terminals heading up corresponds to contravariant indices and terminals heading down corresponds to covariant indices. Invariant tensors are denoted as follows: (Euclidean) metric tensor and its inverse δ_{ij} = and δ^{ij} = (cup), Kronecker delta δ^i_j = (line), the metric volume form and its index-raised version ε_{ijk} = and ε^{ijk} =. We follow Penrose[9] for the design of ε_{ijk} and ε^{ijk}. For three-dimensional Euclidean space, ε_{ijk} and ε^{ijk} can be "heterarchized," i.e., we can substitute (diagram) $\to$ (diagram) and (diagram) $\to -$(diagram). A rank-2 contravariant tensor, a $\binom{2}{0}$-tensor, X^{ij}, transforms as (diagram) $\to$ (diagram). If the two indices are lowered by the metric, we have a rank-2 contravariant tensor, a $\binom{0}{2}$-tensor, X_{ij}: (diagram) $\to$ (diagram). Contravariant indices get dressed with upward arrowheads, and covariant indices get dressed with downward arrowheads. Graphical calculus with this "up-down hierarchy" indeed has a beautiful and neat syntax; however, we decided not to unnecessarily pursue the distinction between covariant and contravariant indices which might confuse the readers who are mathematically unsophisticated. Penrose[9], for example, will be helpful to interested readers.[10]

P51 [$\star$] Based on our definition of the inertia tensor, the tensor connecting two vectors $\vec{L}$ and $\vec{\omega}$, what kind of rank-2 tensor should the inertia tensor be, among $\binom{2}{0}$, $\binom{1}{1}$, and $\binom{0}{2}$?

P52 [$\star$] Write Eq. (3), (diagram) = (diagram) $-$ (diagram), (diagram) $= -2$ (diagram), and (diagram) $= -6$ in the graphical notation that distinguishes contravariant and covariant indices.

4. Rotation Matrix

The previous section studied how tensors get dressed with rotation matrices, $R_{ij} =$, when rotated. Now, let us focus on the rotation matrix itself. Denote the rotation matrix that encodes a rotation with respect to an axis, specified by a unit vector $\vec{n}$, with an angle α as .

P53 (a) For infinitesimal ϵ, show that $=$ $+ \epsilon$ up to $\mathcal{O}(\epsilon^1)$.

(b) Calculate the commutator between infinitesimal rotation matrices, i.e., $-$, up to $\mathcal{O}(\epsilon^2)$.

P54 [$\star$] is called the generator of rotation.

(a) It is a rank-2 tensor—what representation does it live in? [*Answer*: *rank-2, spin-1*]

(b) Show that $-$ $-$ $-$ is proportional to Kronecker delta. The proportionality constant is called the quadratic Casimir.

P55 [$\star$] Generally, a tensor is called (minus of) the Hodge dual of a vector $\vec{A}$. Let us study its algebra. Calculate the following. (tr of a two-terminal diagram means joining the terminals by Kronecker delta: taking a trace.)

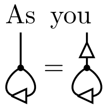
(a) tr ( ), tr ( ), tr ( )

(b) $(\;)^k :=$

P56 [$\star$] Now, let us exponentiate the infinitesimal rotation (rotation algebra) to obtain non-infinitesimal ones (rotation group); $= \lim_{N\to\infty} (\;)^N = \exp(\alpha\;)$. Calculate this and obtain the Rodrigues' rotation formula.

P57 [$\star$] Given a rotation matrix R_{ij}, how can one figure out its rotation angle and rotation axis?

(a) Show that tr $(\;) = 1 + 2\cos\alpha$.

(b) As you might discovered while playing with **P45**,

. What does this mean?

(c) The previous two problems are related to the spin-0 and spin-1 parts of , respectively. What is its spin-2 part?

5. Finding the Oblivious Delta Term

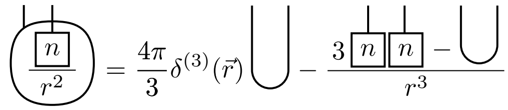
In electromagnetism, the fields that Gilbert and Ampère (or, electric and magnetic) dipole moments (denoted $\vec{p}$ and $\vec{m}$ below) generate contain delta function terms as the following.

$$-\nabla\left(\frac{\vec{p}\cdot\vec{n}}{4\pi r^2}\right) = \frac{3\,\vec{n}\,\vec{n}\cdot\vec{p}-\vec{p}}{4\pi r^3} - \frac{1}{3}\vec{p}\,\delta^{(3)}(\vec{r}) \tag{27}$$

$$\nabla\times\left(\frac{\vec{m}\times\vec{n}}{4\pi r^2}\right) = \frac{3\,\vec{n}\,\vec{n}\cdot\vec{m}-\vec{m}}{4\pi r^3} + \frac{2}{3}\vec{m}\,\delta^{(3)}(\vec{r}) \tag{28}$$

Undergraduate education used to avoid tensorial expressions so that somewhat long approaches are required to derive these identities.[13,14] However, tensor calculus allows one to algebraically obtain the delta function terms in dipolar fields, and we discuss its graphical form.

P58 [$\star$] Show that applying the decomposition Eq. (26) to $\partial_i(n_j/r^2)$ reads as the following.


(29)

P59 Derive Eq. (27) and Eq. (28) from Eq. (29).

P60 [$\star$] Consider a stationary flow $\vec{u}(\vec{r})$ of an incompressible ($\nabla\cdot\vec{u} = 0$), viscous fluid at low Reynolds number. The Navier-Stokes equation reads $\mu\nabla^2\vec{u} - \nabla p = 0$. A "unit" nonhomogeneous solution is called Stokeslet: $\mu\nabla^2\vec{u}-\nabla p = \vec{f}\delta^{(3)}(\vec{r})$.

(a) Show that the pressure field $p(\vec{r})$ is determined before knowing $\vec{u}(\vec{r})$ as $p(\vec{r}) = p_0 + \vec{f}\cdot\nabla\frac{1}{4\pi r}$, assuming that $p(\vec{r}) \to p_0$ as $r\to\infty$.

(b) Show that $\vec{u}(\vec{r}) = -\frac{1}{8\pi\mu r}(\vec{f}+\vec{n}\,\vec{n}\cdot\vec{f})$. [*Hint:* **P32**a.]

6. Harmonic Tensor Fields

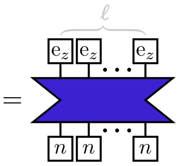
This section is entirely in the [$\star$] level.

Eq. (29) is related to dipolar fields; its generalization to arbitrary ranks is related to multipolar fields, which are described by the spherical harmonics $\mathrm{Y}_\ell^m(\theta,\phi)$. Consider describing scalar Laplace's equation $\nabla^2 f(\vec{r}) = 0$ in spherical coordinates. Spherical symmetry tells us that the space of solutions is spanned by functions of the form $R_\ell(r)\mathrm{Y}_\ell^m(\theta,\phi)$ with $\ell = 0,1,2,\cdots$ and $m = -\ell, \ell+1, \cdots + \ell$.

$$0 = \frac{1}{R_\ell(r)}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}R_\ell(r) + \frac{1}{r^2}\frac{1}{\mathrm{Y}_\ell^m(\theta,\phi)}\nabla\!\!\!\!\nabla^2\mathrm{Y}_\ell^m(\theta,\phi) \tag{30}$$

Here, we defined $\nabla\!\!\!\!\nabla^2$ as Laplacian on unit sphere. Since $-\nabla\!\!\!\!\nabla^2\mathrm{Y}_\ell^m(\theta,\phi) = \ell(\ell+1)\mathrm{Y}_\ell^m(\theta,\phi)$, $R_\ell(r) = r^\ell$ or $1/r^{\ell+1}$. The former gives solid harmonics that are not singular at the origin ("regular" solid harmonics).

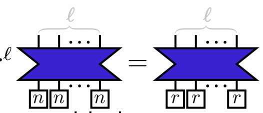
While dealing with spherical harmonics, working in Cartesian coordinates can provide shortcuts in calculations, so we will develop such Cartesian technology in several following problems. From dimensional considerations, we expect that $r^\ell\mathrm{Y}_\ell^m(\theta,\phi)$ can be written as a ℓ^{th}-order polynomial of z, x, and y, which are $\vec{r}\cdot\vec{\mathrm{e}}_z$, $\vec{r}\cdot\vec{\mathrm{e}}_x$, and $\vec{r}\cdot\vec{\mathrm{e}}_y$, respectively. This means that there exists a converter (a bunch of Kronecker deltas) that connects ℓ $\vec{r}$'s with $\vec{\mathrm{e}}_z$'s, $\vec{\mathrm{e}}_x$'s, and $\vec{\mathrm{e}}_y$'s of total number ℓ so that such contractions make $r^\ell\mathrm{Y}_\ell^m(\theta,\phi)$. Denote this "converter" as . For example, $r^\ell\mathrm{Y}_\ell^0(\theta,\phi) \propto r^\ell\mathrm{P}_\ell(\vec{n}\cdot\vec{\mathrm{e}}_z)$ has no ϕ dependence so that $\vec{n}$ and $\vec{\mathrm{e}}_z$ are the only players; therefore, define by Eq. (31). Then, allowing $\vec{\mathrm{e}}_x$ and $\vec{\mathrm{e}}_y$ to join the game, we get the spherical harmonics algebraically, rather than the analytic approach—solving partial differential equation on the sphere for nonzero m.

$\mathrm{P}_\ell(\vec{n}\cdot\vec{\mathrm{e}}_z) =$

. (31)

However, Eq. (31) does not determine uniquely: an antisymmetric component vanishes when identical vectors are contracted in its indices. Thus, we demand the upper indices to be totally symmetric and so do the lower indices in addition to Eq. (31). For example, $P_1(\vec{n}\cdot\vec{n}') = \vec{n}\cdot\vec{n}' \implies$ [diagram] $=$ [diagram], and $P_2(\vec{n}\cdot\vec{n}') = \frac{3}{2}(\vec{n}\cdot\vec{n}')(\vec{n}\cdot\vec{n}') - \frac{1}{2}(\vec{n}\cdot\vec{n})(\vec{n}'\cdot\vec{n}') \implies$ [diagram] $= \frac{3}{2}$ [diagram] $- \frac{1}{2}$ [diagram]. Note that $\frac{3}{2}$ [diagram] $- \frac{1}{2}$ [diagram], $\frac{3}{2}$ [diagram] $- \frac{1}{2}$ [diagram], or other possibilities such as $\frac{1.7}{2}$ [diagram] $+ \frac{1.3}{2}$ [diagram] $- \frac{1}{2}$ [diagram] also gives $\frac{3}{2}(\vec{n}\cdot\vec{n}')^2 - \frac{1}{2}$ when contracted with $\vec{n}$ in its lower indices and $\vec{n}'$ in its upper indices but we choose "the" totally symmetric one, $\frac{3}{2}$ [diagram] $- \frac{1}{2}$ [diagram].

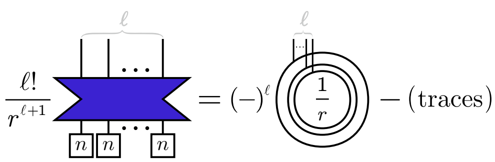
P61 The converter, [diagram], is denoted by a horizontally symmetrical shape. Can you explain why would the designer choose such shape?

P62 [⋆] Taking Laplacian to r^ℓ



should give zero. From this, show that [diagram] is "traceless," i.e., it vanishes when any of its two lower indices are contracted (the same applies to the upper indices).

P63 [⋆] From the generating function of Legendre polynomials (the multipole expansion), prove the following. By "−(traces)," we mean subtracting terms proportional to ⋂ and ⋃ so that the entire right hand side is traceless in the upper indices and so do in the lower indices.



$$\frac{\ell!}{r^{\ell+1}}\ [\text{diagram}] = (-)^\ell\ [\text{diagram}] - (\text{traces}) \qquad (32)$$

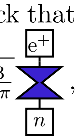
P64 [⋆] So far, we have found that [diagram], regarding only its upper indices, is a rank-ℓ totally symmetric and traceless tensor (i.e., it lives in the spin-ℓ representation of SO(3)). The same applies to the lower indices.

(a) Check that tracelessness holds for $\ell = 2$ and 3 by explicit calculation, given that $P_2(\xi) = \frac{3}{2}\xi^2 - \frac{1}{2}$ and $P_3(\xi) = \frac{5}{2}\xi^3 - \frac{3}{2}\xi$.

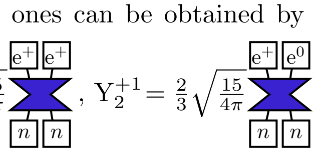
(b) Also, do the inverse direction: find the coefficients a_j where $P_4(\xi) = \sum_{j=0}^4 a_j \xi^j$, using the fact that [diagram] is a rank-4 totally symmetric and traceless tensor. (Fix the overall coefficient by $P_4(1) = 1$.)

Note that taking Laplacian to Eq. (32) gives the equality between $\nabla^2\big(r^{-\ell-1}Y_\ell^m(\vec{n})\big)$ and an ℓ^{th}-order, $(\ell-2)^{\text{th}}$-order, $\cdots$ derivatives of delta function, i.e., point multipole sources. Thus, $\nabla^2\big(r^{-\ell-1}Y_\ell^m(\vec{n})\big)$ is not zero in physicists' sense but raises multipolar singularities at the origin (it is "almost zero").

P65 [⋆] Define $\vec{e}^+ := -\frac{1}{\sqrt{2}}(\vec{e}_x + i\vec{e}_y)$, $\vec{e}^- := +\frac{1}{\sqrt{2}}(\vec{e}_x - i\vec{e}_y)$, and $\vec{e}^0 := \vec{e}_z$ (the spherical basis vectors). Comparing withRefer to the spherical harmonics table, and check that the first few ones can be obtained by $Y_1^{+1} = \sqrt{\frac{3}{4\pi}}$



, $Y_2^{+2} = \frac{2}{3}\sqrt{\frac{15}{8\pi}}$ [diagram], $Y_2^{+1} = \frac{2}{3}\sqrt{\frac{15}{4\pi}}$ [diagram], etc.



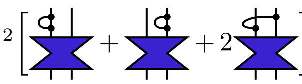
P66 [⋆] $i\hbar$ [diagram] is the "spin operator" (of direction $\vec{e}_i$) acting on vectors. Why?

(a) Show that $(i\hbar)^2$ [diagram] $- (i\hbar)^2$ [diagram] $= (i\hbar)^2$ [diagram]. How does this coincide with the spin operator you have learned in quantum mechanics?

(b) Show that $i\hbar$ [diagram] $= (-)^m\hbar$ [diagram], where $m = +1, 0, -1$. Thus, [diagram]'s are the eigenvectors of z-axis rotation.

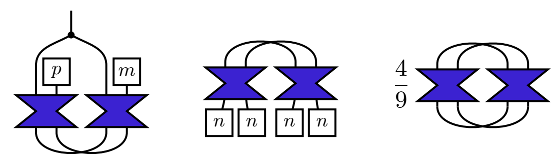
(c) Show that $Y_2^{+1}(\theta,\phi)$ transforms like a vector, particularly, like $\vec{e}^+$, under infinitesimal z-axis rotation.

(d) Calculate $(i\hbar)^2$



and explain its meaning.

P67 [!] Some diagram gymnastics. Calculate the following by diagrams. (You may wonder where these expressions appear. The solution to this problem contains an answer for such curiosity.)



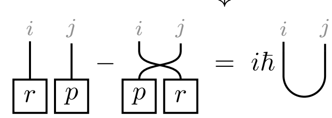
(a) (b) (c)

D. Addendum

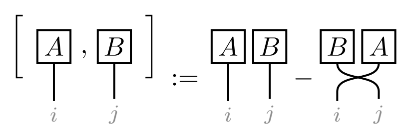
1. Quantum Mechanics with the Graphical Notation

The last application of graphical notation to be mentioned is quantum mechanics. In quantum mechanics, physical quantities are promoted to non-commuting operators. To denote operators in the graphical notation, simply place the boxes in a row and demand the relative positions to be not changed. For example, the canonical commutation relation $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}\hat{1}$, where x_i and p_i are the components of the position vector operator $\vec{r}$ and the momentum vector operator $\vec{p}$, is denoted as follows.

$$\hat{x}_i\hat{p}_j - \hat{p}_j\hat{x}_i = i\hbar\delta_{ij}\hat{1}$$
$$\updownarrow \qquad (33)$$

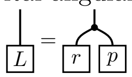


For convenience, introduce the conventional square bracket notation for commutator, instead of inventing a new graphical notation. For example, the commutator of two vector operators $\hat{\vec{A}}$ and $\hat{\vec{B}}$, $[\hat{A}_i, \hat{B}_j] = \hat{A}_i\hat{B}_j - \hat{B}_j\hat{A}_i$, is represented as the following.



$$(34)$$

Remember that by taking a commutator you should only alter the order of boxes, keeping the endpoints of the lines (vector indices) fixed.

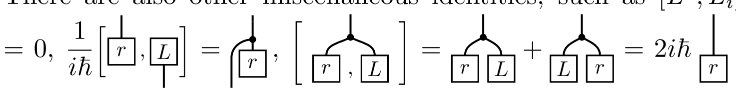
The orbital angular momentum operator is graphically represented as .

P68 Prove the following by diagrams.

(a) $\hat{\vec{r}}\cdot\hat{\vec{p}} = \hat{\vec{p}}\cdot\hat{\vec{r}} + 3i\hbar\hat{1}$ (b) $\hat{\vec{r}}\times\hat{\vec{p}} = -\hat{\vec{p}}\times\hat{\vec{r}}$

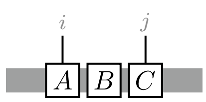
P69 Graphically represent the $\mathsf{so}(3) = \mathsf{su}(2)$ commutation relations, i.e., $[\hat{L}_i, \hat{L}_j] = i\hbar\epsilon_{ijk}\hat{L}_k$. Graphically prove that the orbital angular momentum satisfies such relation.

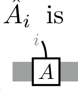
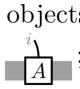
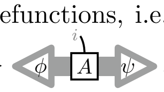
P70 Angular momentum operator identities can be quickly derived using the graphical notation. One of the examples are $\hat{L}^2 = \hat{\vec{L}}\cdot\hat{\vec{L}} = \hat{r}^2\hat{p}^2 - (\hat{\vec{r}}\cdot\hat{\vec{p}})^2 + i\hbar\hat{\vec{r}}\cdot\hat{\vec{p}}$. Prove this identity by diagrams.

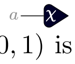
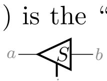
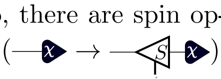
There are also other miscellaneous identities, such as $[\hat{L}^2, \hat{L}_i] = 0$, , $[[\hat{\vec{r}}, \hat{L}^2], \hat{L}^2] = 2\hbar^2(\hat{\vec{r}}\hat{L}^2 + \hat{L}^2\hat{\vec{r}})$, etc. One may benefit from using graphical notation while deriving these identities. We shall omit the details.

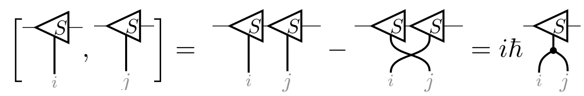
2. Intertwining Different Types of Lines

The graphical notation for operators leads us to an interesting discussion. First, let us hear the designer's alternative choice for denoting noncommutative numbers. Our previous notation can be made more clear by turning on an "order marker," like the following.

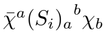
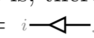
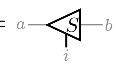
$\hat{A}_i\hat{B}\hat{C}_j \leftrightarrow$  (35)

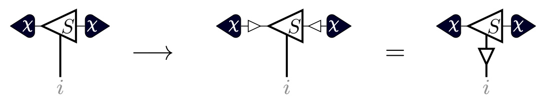
Noncommutativity made explicit: the order of loading boxes on the "conveyor belt" matters. A vector operator $\hat{A}_i$ is a box with a vector tail i and a pair of thick grey lines: . It intertwines two kinds of lines. Note that thick lines are "left-right hierarchical," that is, they must be drawn horizontally and lines stemming left and right are distinguished (cf. "up-down hierarchy" in Section I C 3). Meanwhile, what is that grey conveyor belt anyway? The answer can be found by inferring what objects with a thick grey line represent. They are transformed by objects with one input and one output thick grey lines, such as ; they are wavefunctions, i.e., Hilbert space vectors! For example, $\langle\phi|\,\hat{A}_i\,|\psi\rangle \leftrightarrow$ .

Vectors living in the Hilbert space $\mathcal{H}$ are represented as the objects that have one thick grey line. The reason for choosing a thick line is to imply that it is infinite-dimensional, i.e., wavefunctions $\psi(\vec{r})$ living in $\mathcal{H}$ are labelled by a "continuous index" $\vec{r}$. Meanwhile, we also have seen finite-dimensional Hilbert spaces in undergraduate quantum mechanics, such as spin states. For example, the spin part of an electron wavefunction lives in a Hilbert space of complex dimension two. The state vectors are called spinors: $\chi_a =$  $\in \mathcal{H}$ $(a = 0, 1)$. $(\chi_0, \chi_1) = (1, 0)$ is the "up" state, while $(0, 1)$ is the "down" state. Also, there are spin operators ${(S_i)_a}^b =$  acting on spinors (), where $i = 1, 2, 3$. The index i here is an $\mathsf{SO}(3)$-index we have been always discussing about. The spin operators satisfy the following commutation relations.

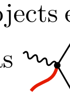
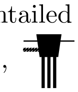
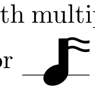
 (36)

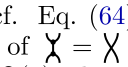
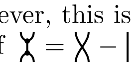
As usual, Pauli matrices can be employed to represent the spin operators as ${(S_i)_a}^b = \frac{1}{2}{(\sigma_i)_a}^b$.

Note that $\mathcal{H}$-lines (spin-1/2 lines, labelled by a, b, $\cdots$) are being differently denoted from $\mathsf{so}(3)$-lines (spin-1 lines, labelled by i, j, $\cdots$), because the two vector spaces are different. We cannot connect the two ends of a spin-1 line and a spin-1/2 line directly such as . Also, there is no invertible "basis change" matrix that connects one spin-1 and one spin-1/2 line. Even the dimensions are different—3 and 2. We must use appropriate converters (invariant symbols). What we expect from such converters is a "proper propagation of arrowheads." For example, rotating an $\mathsf{SO}(3)$-vector translates into unitary transformation of the corresponding spinor in the spin-1/2 language; that is, there is a correspondence between a rotation matrix $R_{ij} =$  and a unitary matrix ${U_a}^b =$. The translator in charge is the spin operator ${(S_i)_a}^b =$ .

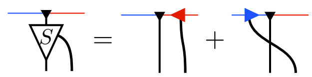


$$\bar{\chi}^a{(S_i)_a}^b\chi_b \longrightarrow \bar{\chi}^c{U^\dagger{}_c}^a{(S_i)_a}^b{U_b}^d\chi_d = R_{ij}\,\bar{\chi}^a{(S_j)_a}^b\chi_b \quad (37)$$

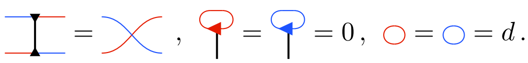
It is not necessary to introduce the concept of representation converters in the context of quantum mechanics (we did because we wanted to start with objects that are familiar from undergraduate education). Objects that intertwines two different kinds of lines naturally appear in group theory: generators of group action, Clebsch-Gordan coefficients, converters between two representations (e.g., differential operators and $\mathsf{SO}(3)$ rotation or spin-3/2 and spin-1/2 representations of $\mathsf{SU}(2)$), etc. Furthermore, it is also possible to consider objects entailed with multiple types and numbers of indices, such as , , or .

In group theory, there exists a systematic procedure of obtaining tensorial identities.[3] We choose not to reproduce it here; most of the essentials of such procedure is already illustrated in a heuristic manner throughout the development of this article. It is all about imposing constraints on the group algebra from demanding the "proper propagation of arrows." For example, the invariance of δ_{ij} under infinitesimal rotations imply that the rotation generators are antisymmetric. The invariance of generators themselves (cf. Eq. (37)) boils down to the commutation relation studied in **P53**b, where ϵ_{ijk} being the structure constant of the $\mathsf{so}(3)$ algebra. The invariance of the structure constant then implies the Jacobi identity (cf. Eq. (64)). One thing that needs an extra care is the analog of  for a general Lie group. In case of the group $\mathsf{SO}(3)$, the invariant symbol ϵ_{ijk}, the rotation generators, and the structure constant all coincide. However, this is not true for general Lie groups so that the analog of  can be not unique.

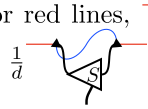
P71 [⋆] In this section, we are broadening our horizons to multiple types of lines. To illustrate the "proper propagation of arrowheads" condition for invariant symbols, consider a hypothetical algebra that has three types of lines: black, red, and blue. Suppose there exists an invariant symbol entailed with all the types of lines: $\sigma^{\mu}_{a\dot{b}} =$ (diagram). The condition for this object to be invariant under infinitesimal transformations is the following (infinitesimal version of) "propagation of arrowheads."


(38)

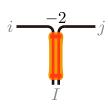
Suppose that the basic grammar of this algebraic system is given by the following.


(39)

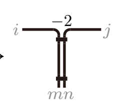
Find the expression for the generator for red lines, (diagram), provided that (diagram) is known. [*Answer*: $\frac{1}{d}$


.]

P72 [!] The idea of collective indices is to regard a bunch of indices as one group. Consider a grouping $I \leftrightarrow [ij]$. Then, an antisymmetric tensor $T_{ij} = T_{[ij]}$ is written "T_I" in the collective index notation. We can regard i and I as "different types of lines" and investigate the correspondence between the i-world and the I-world. It is instructive because one can derive the algebra of I-tensors from the lower level syntax, the algebra of i-tensors. In the graphical notation, the converter between the two worlds is drawn as the following.

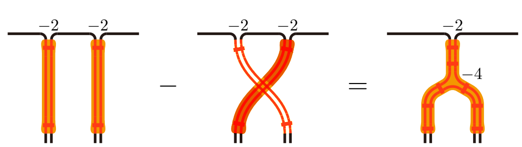
$$(M_I)_{ij} =$$

(40)

Do you see the antisymmetrizer inside the "sheath?" If we peel off the sheath, that is, when $I \leftrightarrow [mn]$,

$$(M_I)_{ij} \leftrightarrow$$

$$= -2\delta_{i[m}\delta_{n]j}. \qquad (41)$$

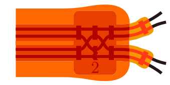
In this problem, i, j, $\cdots$ are $\mathsf{SO}(3)$-vector indices as usual.

(a) Prove the commutation relation $(M_I)_{ik}(M_J)_{kj} - (M_J)_{ik}(M_I)_{kj} = f_{KIJ}(M_K)_{ij}$, where $f_{KIJ} = -2(\delta_{k[m}\delta_{n][r}\delta_{s]l} - \delta_{l[m}\delta_{n][r}\delta_{s]k})$ when $I \leftrightarrow [mn]$, $J \leftrightarrow [rs]$, and $K \leftrightarrow [kl]$.

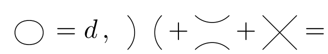

(42)

(b) The I-version of Kronecker delta is the antisymmetric projector; $\delta_{IJ} \leftrightarrow \delta_{m[r}\delta_{s]n}$ when $I \leftrightarrow [mn]$ and $J \leftrightarrow [rs]$. In the graphical notation, $\delta_{IJ} =$ (diagram). This satisfies $\delta_{IJ}\delta_{JK} = \delta_{IK}$. What is the I-version of ϵ_{ijk} then? The requirement is that δ_{IJ} and ϵ_{IJK} have the same syntax with δ_{ij} and ϵ_{ijk}. (After constructing δ_{IJ} and ϵ_{IJK} and establishing their syntax that is derivable from the lower level structures (the i-world), we can solely work on the I-world when dealing with I-world identities, not referring to the "microscopic implementations.") [*Answer*: $i\,\epsilon_{IJK} := \frac{\pm 1}{\sqrt{2}} f_{IJK}$.]

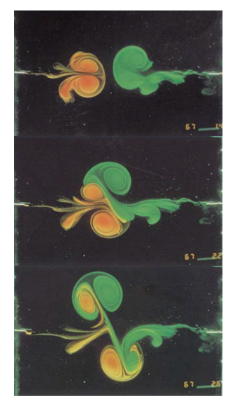
(c) We stacked two i-lines and then made the I-world that reproduces the syntax of its mother. We can repeat this procedure to build, say, the **I**-world, where $\mathbf{I} \leftrightarrow [IJ]$, that has the same syntax with the I-world so with the i-world.


(43)

Furthermore, we can repeat this again and again and construct an infinite tower—fractals! On the other hand, can we step down? An interesting observation by Penrose[17] is that it is possible. What he calls "binors" have the following property.


(44)

The relation between binors and the i-world parallels that of the i-world and the I-world. That is, when $i \leftrightarrow [ab]$ and $j \leftrightarrow [cd]$, $\delta_{ij} =$ (diagram) $\leftrightarrow \delta_{a[c}\delta_{d]b} =$ (diagram). Show that (diagram) $\leftrightarrow \pm\sqrt{2}$ (diagram) reproduces (diagram) $=$ (diagram) $-$ (diagram). In addition, find the value of d that reproduces (diagram) $= -2$ (diagram), i.e., $(\pm\sqrt{2})^2$ (diagram) $= -2$ (diagram). [*Answer*: $d = -2$.]

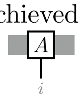
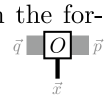


These diagrams for collective indices reminds the author of this picture of a collision between two dipolar vortices.[18] Two vortices interact by interchanging half of their body with each other; a big particle comprised of two smaller elements interact with another one by interchanging one of their constituents. It is analogous to the case of binors where two spin-1/2 "particles" made a spin-1 "particle." When the "small" particles are appropriately grouped, the syntax of the small world reproduces the syntax of the big world. However, the "big syntax" does not necessarily tell everything about the "small syntax." The vortices may not remain in a dipolar shape after collision in some circumstances, for example; there are binor identities that do not correspond to any i-world expressions. The flow from the small to the big world is one-sided; information is lost. This implies a possibility of multiple small worlds corresponding to the big world; a number of "small syntax" can implement a "universal syntax." Repeating the grouping procedure, microscopic details will be completely washed out so that a fixed point will be achieved; recall the "self-similarity" of $\mathsf{SO}(3)$-tensors under antisymmetric grouping. The "renormalization group flow" flows from binors to the i-world which is the fixed point.

3. *Connection to Feynman Diagrams*

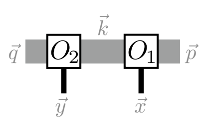
We just have opened up new possibilities to our toy box: vertices that intertwine different kinds of lines. Now, one may be curious about what is going on in Feynman diagrams, where various kinds of lines standing for different kinds of particles appear: straight, wiggly, squiggly, dotted, $\cdots$. Regarding the structural aspects only, Feynman diagrams are just special categories of the graphical equations we have been discussing.

Let us consider Feynman diagrams in continuum quantum field theory as an example. Field theory studies about physical quantities distributed on spacetime, such as electromagnetic potential $(V(t,\vec{x}), \vec{A}(t,\vec{x}))$ or displacement field of vibrating atoms in a solid lattice, $\vec{\xi}(t,\vec{x})$. Typically, field variables take values at an infinite or large number (i.e., an order of Avogadro number) of places—electromagnetic fields take values at every point in spacetime, and the displacement field takes values at each site on the lattice. In this sense, field theory is a study of infinite or large degrees of freedom. It is typical to approximate discrete fields such as $\vec{\xi}(t,\vec{x})$ as continuum fields. For instance, for $\vec{\xi}(t,\vec{x})$, this is done by regarding the atomic spacing to be small. Quantum field theory studies quantum fluctuations of fields. Let $\phi(t,\vec{x})$ be an anonymous field (the letter "ϕ" for degrees of "freedom"). When $\hbar$ is turned on, the value of $\phi(t,\vec{x})$ gets uncertain as a particle's position variables do in quantum mechanics; that is, it becomes an operator, $\hat{\phi}(t,\vec{x})$. The result is various operator fields: $\hat{\phi}^2(t,\vec{x})$, $\hat{\phi}^3(t,\vec{x})$, and so on (cf. $\hat{x}^2$, $\hat{x}^3$, $\cdots$ in quantum mechanics).[19] Generally, ϕ and its conjugate momentum π together comprise all kinds of fields in a theory.

Think of some local operator $\hat{O}(t,\vec{x})$. When the Hamiltonian $\hat{H}$ of the system is given, operators evolve through time according to the time evolution law in the Heisenberg picture: $\hat{O}(t,\vec{x}) = \hat{U}^{-1}(t \leftarrow t_0)\,\hat{O}(t_0,\vec{x})\,\hat{U}(t \leftarrow t_0)$ with $\hat{U}(t \leftarrow t_0)$ being the time-evolution operator $\exp\left(\frac{1}{i\hbar}(t-t_0)\,\hat{H}\right)$. Let us first consider operators at a certain equal-time slice, thus suppressing the time label. How can $\hat{O}(\vec{x})$ be represented graphically? It is achieved by simply replacing the discrete index i in the former  of Section I D 2 into a continuous index $\vec{x}$:  $:= O^{\vec{q}}{}_{\vec{p}}(\vec{x})$. The continuous indices $\vec{p}, \vec{q}, \cdots$ label $\mathcal{H}$-lines. (A "state" in a quantum field theory corresponds to a probability profile of field configurations at a certain time slice, so the state space $\mathcal{H}$ is infinite-dimensional.) Here, instead of being denoted by using the hat symbol, an operator is seen as a tensor that has one input $\mathcal{H}$-line and one output $\mathcal{H}$-line: it takes a state in the Hilbert space and returns another state!

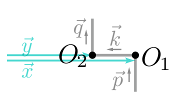
Now, successive application of such operators would be denoted as the following.

$$O_2{}^{\vec{q}}{}_{\vec{k}}(\vec{y})\, O_1{}^{\vec{k}}{}_{\vec{p}}(\vec{x}) \;=\; \tag{45}$$



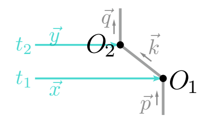
$\vec{k}$ is a dummy index. The designer now pursues minimalism and suggests us to put boxes at position $\vec{x}$ rather than writing "$\vec{x}$" at its side—abolishing the abstraction inherent in the gap between letters and its meaning and returning to the primitive, "hieroglyphic" level.

$$O_2{}^{\vec{q}}{}_{\vec{k}}(\vec{y})\, O_1{}^{\vec{k}}{}_{\vec{p}}(\vec{x}) \;=\; \tag{46}$$



The position vectors $\vec{x}$ and $\vec{y}$ (indicated by mint arrows) are three-dimensional; we regret that we could not use a 3D printer to print the diagram Eq. (46) in three dimensions. Instead of that, we expressed the spatial position as horizontal displacement. We drew small arrows alongside the grey lines to clarify the operator ordering, as the placement of operators now depends on the position labels rather than the operator ordering so that the "right-to-left hierarchy" reading rule cannot be applied anymore.

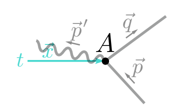
Eq. (46) is in fact an equal-time Feynman diagram! Indices $\vec{p}$, $\vec{k}$, $\vec{q}$ are momentum labels, indicating momentum flows. Now, let us move on to operator fields at different times. A diagram corresponding to an expression $O_2{}^{\vec{q}}{}_{\vec{k}}(t_2,\vec{y})O_1{}^{\vec{k}}{}_{\vec{p}}(t_1,\vec{x})$, for $t_2 > t_1$, would be the following.



(47)

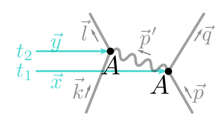
As conventional space-time diagrams do, the flow of time is depicted as vertical direction, while the spatial dimension is depicted horizontally. This diagram shows a particle moving in spacetime. Recall that the total space of invariants was spanned by rank-0, 1, 2, $\cdots$ tensors in the case of finite-dimensional tensors. Analogously, the whole possibilities of quantum fluctuations come in zero-particle state, one-particle states, two-particle states, and so on: diagrams with 0, 1, 2, $\cdots$ lines.

Finally, if another type of Hilbert space $\mathcal{H}'$ (labelled by primed indices $\vec{p}\,'$, $\vec{q}\,'$, $\cdots$) participates in this game, a different kind of grey line comes into play. For example, there can be an operator, say, $A^{\vec{p}\,'\vec{q}}{}_{\vec{p}}(t,\vec{x})$, that intertwines two $\mathcal{H}$ indices and one $\mathcal{H}'$ index:



. (48)

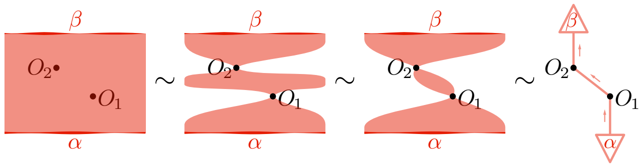
Then, we can assemble such operators to build diagrams such as



, (49)

which denotes $A_{\vec{p}\,'}{}^{\vec{l}}{}_{\vec{k}}(t_2,\vec{y})A^{\vec{p}\,'\vec{q}}{}_{\vec{p}}(t_1,\vec{x})$: two "——" particles are scattered via exchanging a "∿∿∿" particle. Different types of Hilbert spaces means different kinds of particles. $A^{\vec{p}\,'\vec{q}}{}_{\vec{p}}(t,\vec{x})$, dressed with various types of indices, functions as a converter between Hilbert spaces: an interaction vertex between particles. The lines are particles: objects, nouns. The vertices are interactions: morphisms, verbs. Eq. (49) is a typical diagram you would meet in quantum electrodynamics, up to extra structures including polarization and spinor information.

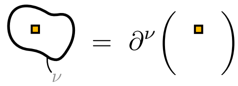
The significance of Feynman diagrams in physics is perhaps that it explicates the particle interpretation. The deduction of Feynman diagrams above may be vague in the standards of physics. For example, are those grey lines really particles, while they come from the graphical representation of operator fields $\hat{O}_1(t,\vec{x})$, $\hat{O}_2(t,\vec{x})$, $\cdots$ that are arbitrary? This is because the discussion was somewhat formal. A typical explanation is opposite to discussions here: one arrives at birdtracks from Feynman diagrams by leaving only the group-theoretical part. To make the physical content clearer, it is recommended to start with concrete physical examples, e.g., analyzing the grammar of a Feynman diagram in quantum mechanics. Studying how Feynman diagrams arise in the Hamiltonian framework of quantum field theory is also helpful. Further, Lagrangian (functional) formalism[20] helps to comprehend the "input and output" nature of Feynman diagram elements. The worldline formalism will also provide insights by directly associating the lines in Feynman diagrams with first-quantized Hilbert space. Depending on the formalism, the specific interpretation of Feynman diagrams may appear a little differently. For example, the diagram Eq. (47) can be seen as depicting a path integration with operator insertions $\hat{O}_1(t,\vec{x})$ and $\hat{O}_2(t,\vec{x})$ presented in a form of "gluing formula," rather than a particle process; Eq. (50) schematically shows this idea, where red shade represents path integration and the bumps at the ends depict the boundary conditions of the path integral.



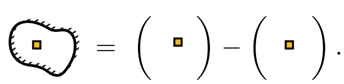
(50)

Starting from diagrams of finite-dimensional tensors, our graphical calculus has undergone a rapid expansion in the past two sections: the addition of infinite-dimensional lines and intertwiners carrying multiple kinds of lines. What is next? Following purely graphical reasoning, one can think of braided diagrams (where there exists a notion of "passing over" and "passing under" when two lines meet), ribbon diagrams (lines become stiff ribbons so that they can be twisted), "straw" diagrams (a cross-section of lines become a loop rather than a point), two-categories (faces also comes into play in addition to vertices and lines), and so on. In fact, the first three appear when considering anyons, non-commutative field theory, and string theory. However, in general, whether such extended diagrammatic systems have connections with actual physics is not certain. Some exotic diagrammatic systems may fail to find physical significance yet remaining as a mathematical possibility. Whether an extended diagrammatic syntax can be derivable from some Lagrangian is also not certain, yet a "radical" scenario is to build a quantum field theory without reference to Lagrangian by taking the "diagrammar" as its definition. Nonetheless, there are always rooms for extended objects.

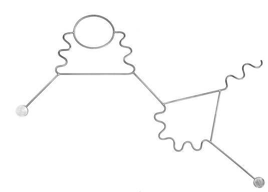
One interesting extended graphical element that comes from standard quantum field theory is the topological surface operator. Let us demonstrate it briefly in the case of Lorentz-covariant scalar field theory. Classically, energy-momentum is locally conserved: $\partial_\mu T^{\mu\nu}[\phi](x) = 0$, where $T^{\mu\nu}[\phi](x)$ is the stress-energy tensor for a field $\phi(x)$. This leads to a global conservation: $-\int d^3x\, T^{0\nu}[\phi](x^0,\vec{x})$ is a conserved quantity (remains the same as x^0 changes). In fact, $P^\nu[\phi](\mathcal{S}) := -\int_{\mathcal{S}} d^3\Sigma_\mu\, T^{\mu\nu}[\phi](x)$ for an arbitrary exact hypersurface $\mathcal{S}$ equals to zero. Now, the quantum version of the local conservation of energy-momentum is the Ward identity[22] $\frac{\partial}{\partial x^\mu}\langle T^{\mu\nu}(x) O_1(x_1)\cdots O_n(x_n)\rangle = \sum_{j=1}^n -\delta^{(4)}(x-x_n)\frac{\partial}{\partial x_{j\nu}}\langle O_1(x_1)\cdots O_n(x_n)\rangle$, where the symbol $\langle\ \rangle$ stands for averaging over all quantum fluctuations. Then, it turns out that $\langle P^\nu(\partial\mathcal{W}) O(x)\rangle = \partial^\nu\langle O(x)\rangle$ holds for any volume $\mathcal{W}$ that contains x. (When multiple operators are inserted, the differentiation only applies for operators put inside $\mathcal{W}$.) To "draw" this identity, we have the following.


(51)

Wait, this is the "differentiation balloon!" (If you feel the tail "$\searrow_\nu$" somewhat artificial, we could have drawn the picture more "descriptively":


(52)

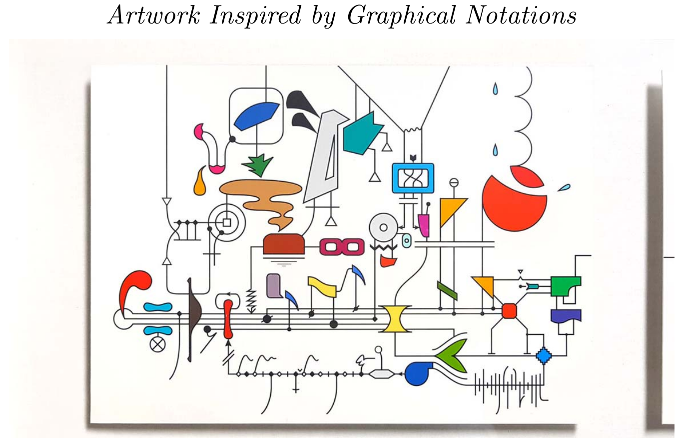
The "hairs" on the surface are the Killing vector field $\xi^\mu\partial_\mu = \delta^\mu_\nu\partial_\mu$. If we were considering rotations $\xi^\mu\partial_\mu = -2(x_{[\alpha}\delta^\mu_{\beta]})\partial_\mu$, the hairs would spiral around the origin. We believe that the abstraction of the hairs into "$\searrow_\nu$" is acceptable enough.) We just "physically implemented" or "materialized"[23] the differentiation balloon notation by a topological surface operator. Such an inversion is intriguing. A Feynman diagram notates an S-matrix that corresponds to a scattering scenario. At the same time, scattering of particles itself also "notates" the Feynman diagram; it implements the choreography (computes the diagram). Drawing hands.[24] Mathematical structures "represent" the physical reality and vice versa. Physical reality, or materials, also serves as a notation. Among various notations, notations that are in the primitive, "hieroglyphic" level may be likely to descriptively display the physical reality—if we assume that the physical reality is written in geometric terms. Or, to be more conservative, graphical notations provide a way to give a physical interpretation of mathematical structures of a theory (the "notational realism"). Think of the way how Feynman diagrams give an answer to a question such as "What is a photon?"


Edward Tufte, *Torqued Space-Time Feynman diagram*, 2012. Stainless steel.

A final comment: having acquainted with group theory before meeting Feynman diagrams, we can ask a question: can we continue the habits from group theory to quantum field theory? Can we make a parallel between the two? Can we study the interactions of particles by applying the systematic procedure of demanding "proper propagation of arrows?" It seems that modern amplitude techniques implement this idea. In the story of color-kinematics duality,[38,39] it turns out that the kinematic factors of a Feynman diagram satisfy a relation analogous to the Jacobi identity. Moreover, this can be explicated by carrying out the idea of "S-matrices as momentum-index tensors" as a diffeomorphism Lie algebra for self-dual Yang-Mills theory.[40–43] In the spinor-helicity formalism,[37] scattering amplitudes are investigated from symmetry principles. Imagine if you can touch a Feynman diagram (should be felt like a steel wire or something else). Grab one of its arms and twist it one direction as if you are twiddling a knob. Then, the amplitude somehow "reacts" to this transformation (called the little group action). Doing a kind of dimensional analysis on the little group weight of each arm, you can deduce the form of the scattering amplitude! It would be interesting to see how these approaches will change our understandings on quantum field theory.

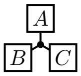
Bonus Cuts

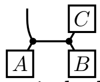
Artwork Inspired by Graphical Notations

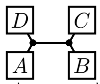

Joon-Hwi Kim, *A Pleasant Dream II*, 2016.
Digital printing on canvas.

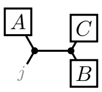
II. Solutions

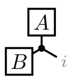
A. Graphical Vector Algebra

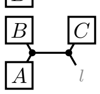
A1 (a) . Depending on grouping the three boxes, you can read off this diagram various ways, such as $\vec{B}\cdot(\vec{C}\times\vec{A})$ and $\vec{C}\cdot(\vec{A}\times\vec{B})$. Reading in clockwise direction gives $-\vec{A}\times(\vec{C}\times\vec{B})$, $-\vec{B}\times(\vec{A}\times\vec{C})$, and $-\vec{C}\times(\vec{B}\times\vec{A})$.

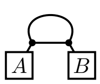
(b) . Reading the two cross product machines anti-clockwise or clockwise, this can be read off by $\vec{A}\times(\vec{B}\times\vec{C})$, $-(\vec{B}\times\vec{C})\times\vec{A}$, $-\vec{A}\times(\vec{C}\times\vec{B})$, and $(\vec{C}\times\vec{B})\times\vec{A}$.

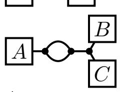
(c) . Different groupings give different readings. $\vec{A}$ and $\vec{D}$ versus $\vec{B}$ and $\vec{C}$ gives $(\vec{D}\times\vec{A})\cdot(\vec{B}\times\vec{C})$. $\vec{A}$ versus the others gives $\vec{A}\cdot\big((\vec{B}\times\vec{C})\times\vec{D}\big)$. $\vec{B}$ versus the others, $\vec{C}$ versus the others, and $\vec{D}$ versus the others are also possible. In addition, reading the cross product machines anti-clockwise or clockwise gives further disguises of an identical expression.

A2 (a) ; $\epsilon_{jlk}A_k(-\epsilon_{lmn}C_mB_n)$.

(b) ; $A_lB_k\epsilon_{lki}$.

(c) ; $B_i(-\epsilon_{ijk}(\epsilon_{jlm}C_m)A_k)$.

(d) ; $(-\epsilon_{ijl}A_l)(-\epsilon_{jkm}B_m)\delta_{ki}$.

(e) ;

$A_i\epsilon_{imn}\epsilon_{jnm}\epsilon_{jkl}C_kB_l$, $\epsilon_{ijk}C_iB_j(\epsilon_{klm}\epsilon_{mln}A_n)$, etc.

A3 (a)  $= A_iB_j$.

(b) $A_kB_k\delta_{ij}$, labelling the two ends of the Kronecker delta i and j, respectively.

(c) $A_iB_jC_k\epsilon_{ijk}$ or $\vec{A}\cdot(\vec{B}\times\vec{C})$, etc.

(d) $A_iB_l\epsilon_{ijk}\epsilon_{ljk}$, etc.

(e) i. $A_jB_kC_mD_nE_o\epsilon_{ijk}\epsilon_{iol}\epsilon_{lmn}$, or $[((\vec{A}\times\vec{B})\times\vec{E})\times\vec{C}]\cdot\vec{D}$, $\vec{E}\cdot\Big((\vec{C}\times\vec{D})\times(\vec{A}\times\vec{B})\Big)$, etc.

ii. $A_jC_mD_nE_o\epsilon_{ijk}\epsilon_{iol}\epsilon_{lmn}$, labelling the end of the diagram by k. Or, $(\vec{E}\times(\vec{C}\times\vec{D}))\times\vec{A}$, etc.

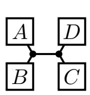
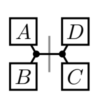
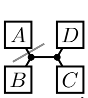
iii. $A_jB_kC_mD_n\epsilon_{ijk}\epsilon_{iol}\epsilon_{lmn}$, labelling the end of the diagram by o. Or, $(\vec{C}\times\vec{D})\times(\vec{A}\times\vec{B})$, etc.

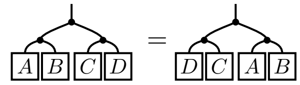
(f) $\delta_{ij}\delta_{ij}$, δ_{ii}, etc.

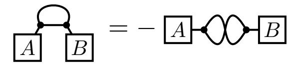
(g) $\epsilon_{ijk}\delta_{jk}$, labelling the end of the diagram by i.

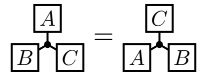
(h) $\epsilon_{ijk}(-\epsilon_{ijk})$, etc.

(i) $\epsilon_{ijk}\epsilon_{jki}$, etc.

A4 Different groupings of a diagram ,  and , gives $(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D})$ and $\vec{A}\cdot(\vec{B}\times(\vec{C}\times\vec{D}))$, respectively.

A5 (a) .

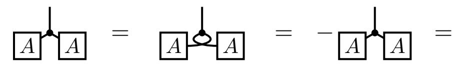
(b) .

(c) .

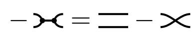
A6 (a) $\vec{A}\times(\vec{B}\times\vec{C})+\vec{B}\times(\vec{C}\times\vec{A})+\vec{C}\times(\vec{A}\times\vec{B})=0$.

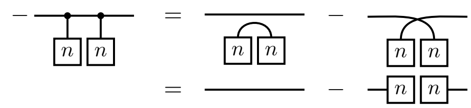
(b) $\epsilon_{imj}A_m\epsilon_{jnk}B_n = A_kB_i - \delta_{ik}A_lB_l$.

A7 In the graphical notation, do the "clank" procedure. In the plaintext notation, rearrange terms, relabel dummy indices, and then permute two indices of the epsilon tensor.

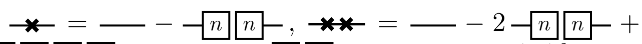


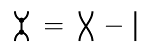
$$\epsilon_{ijk}A_jA_k = \begin{matrix}\epsilon_{ijk}A_kA_j\\ \shortparallel \\ \epsilon_{ikj}A_jA_k\end{matrix} = -\epsilon_{ijk}A_jA_k = 0 \tag{53}$$

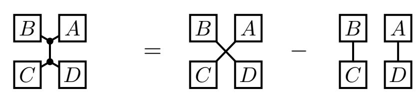
A8 Use  and $\vec{n}\cdot\vec{n}=1$.



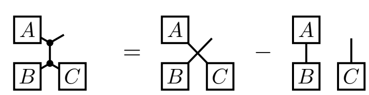
$$\begin{aligned}-\epsilon_{ilk}n_l\epsilon_{kmj}n_m &= \delta_{ij}\delta_{lm}n_ln_m - \delta_{im}\delta_{lj}n_ln_m\\ &= \delta_{ij} - n_in_j\end{aligned} \tag{54}$$

A9 As  ; is idempotent.

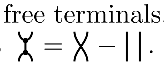
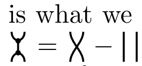
A10 From the given "bones" , we find Eqs. (55) and (56).

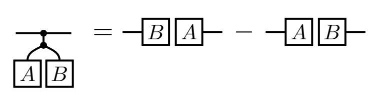


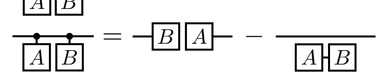
$$(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = \vec{A}\cdot\vec{C}\,\vec{B}\cdot\vec{D} - \vec{A}\cdot\vec{D}\,\vec{B}\cdot\vec{C} \tag{55}$$



$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B}) \tag{56}$$

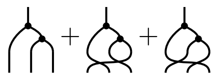
Eq. (55) is a scalar equation, while Eq. (56) is a vector equation. We can "peel off" to obtain identities of two, three, and four free terminals. The resulting four-terminal equation is just . The three-terminal equation is what we get when a single $\vec{A}$ at one of the terminals of  is attached, which is not that interesting. In case of two-terminal equations, there are two possibilities.

 (57)

 (58)

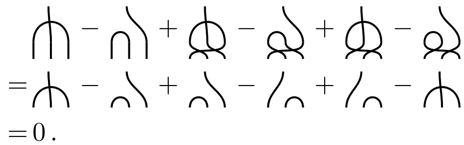
The former relates antisymmetrization with epsilon tensors. The later describes matrix product of Hodge dual of two vectors.

A11 (a) Extract the "bones" of the Jacobi identity. Then, what we have to prove is that



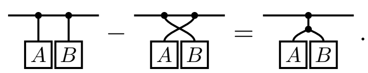
(59)

is equal to zero. Following the instruction given in the problem, we find



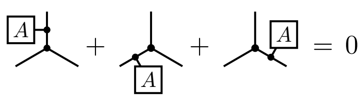
(60) (61) (62)

(b) We can obtain zero, one, two, and three-terminal identities by attaching the "flesh pieces." The one-terminal identity is **P6a** itself, that is, $\vec{A}\times(\vec{B}\times\vec{C})+\vec{B}\times(\vec{C}\times\vec{A})+\vec{C}\times(\vec{A}\times\vec{B})=0$. Contracting this with a vector $\vec{D}$ gives the zero-terminal identity: $\vec{A}\cdot\Big((\vec{B}\times\vec{C})\times\vec{D}\Big)+\vec{B}\cdot\Big((\vec{C}\times\vec{A})\times\vec{D}\Big)+\vec{C}\cdot\Big((\vec{A}\times\vec{B})\times\vec{D}\Big)=0$. In case of two-terminal identity, it is written graphically as



(63)

You will later meet this identity again: this is the commutation relation of the $\mathfrak{so}(3)$ algebra. The three-terminal identity reads as follows.



(64)

This will be interpreted as the invariance of ϵ_{ijk} with respect to infinitesimal rotation.

A12 (a) = − = −2 (this identity is in fact mentioned in the main article). Note that self-contracting a cross product machine gives a trivial identity, $0=0$.

(b) = − $= 3-3^2=-6$.

(c) Starting from $\epsilon_{ijk}\epsilon_{mnk}=\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm}$, we found that $\epsilon_{ijk}\epsilon_{mjk}=\delta_{im}\delta_{jj}-\delta_{ij}\delta_{jm}=3\delta_{im}-\delta_{im}=2\delta_{im}$ and $\epsilon_{ijk}\epsilon_{ijk}=2\delta_{ii}=6$.

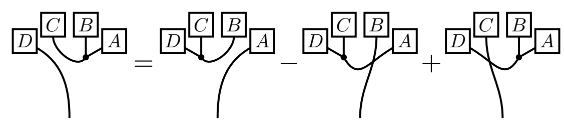
A13 The original purpose of this problem is to let students exploratively find tensor identities while playing with diagrams; however, we approach this problem in a top-down manner. Consider three cross product machines gathered: . We would like to assemble these in a single connected piece. Let the number of connecting lines between two different cores to be a, b, and c, respectively. Surely, any of $a+b$, $b+c$, and $c+a$ should not exceed three, because one cross product machine only has three arms. Without loss of generality, let $a\geq b\geq c$. Then, the possible combinations are $(a,b,c)=(3,0,0)$, $(2,1,1)$, $(2,1,0)$, $(2,0,0)$, $(1,1,0)$, $(1,0,0)$, and $(0,0,0)$: , , , ⋯ . (There can be also self-connections other than mutual connections: such as , etc. However, self-connections always give zero, so they are not our interest.) Rule out $(3,0,0)$, $(1,0,0)$, and $(0,0,0)$, because they are disconnected. Rule out also $(2,1,1)$, because it is equal to zero. Now, the remaining ones are , , and .

A14 and are one-loop diagrams; is a tree-level diagram.

A15 Tree-level diagrams cannot be a 1PI diagram. Also, it can be easily checked that all epsilon networks that have self-connections are not 1PI diagrams. Therefore, among the connected nonzero diagrams we have obtained in **P13**, we only have as a 1PI diagram. On the other hand, $(2,1,1)$, , which vanishes (equals zero) so that we have excluded in **P13**, is also a 1PI diagram. Thus, the answer is and . The first one reduces into − as = − = − . The later reduces into 0, as it is equal to − $= 0-0$.

A16 − = − = .

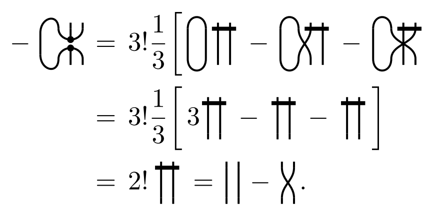
A17 From = − = − , we obtain − + − + = 0. As we have practiced in Section I A 2, we can obtain zero to five terminal identities by attaching the "flesh pieces." The most convenient to denote in the plaintext notation is zero and one-terminal identities. The one-terminal identity reads as follows.



(65)

This is translated to the plaintext notation as $(\vec{A}\cdot\vec{B}\times\vec{C})\vec{D}=\vec{A}(\vec{B}\cdot\vec{C}\times\vec{D})-\vec{B}(\vec{A}\cdot\vec{C}\times\vec{D})+\vec{C}(\vec{A}\cdot\vec{B}\times\vec{D})$. Contracting this with another vector $\vec{E}$, we obtain also a zero-terminal identity: $(\vec{D}\cdot\vec{E})(\vec{A}\cdot\vec{B}\times\vec{C})=(\vec{A}\cdot\vec{E})(\vec{B}\cdot\vec{C}\times\vec{D})-(\vec{B}\cdot\vec{E})(\vec{A}\cdot\vec{C}\times\vec{D})+(\vec{C}\cdot\vec{E})(\vec{A}\cdot\vec{B}\times\vec{D})$.

A18 Using diagrams,

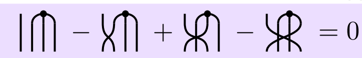


(66) (67) (68)

In the plaintext notation, $\epsilon_{kmn}\epsilon^{kij}=3!\delta^k_{[k}\delta^i_m\delta^j_{n]}=3!\frac{1}{3}(\delta^k_k\delta^i_{[m}\delta^j_{n]}+\delta^k_m\delta^i_{[n}\delta^j_{k]}+\delta^k_n\delta^i_{[k}\delta^j_{m]})=3!\frac{1}{3}(3\delta^i_{[m}\delta^j_{n]}+\delta^i_{[n}\delta^j_{m]}+\delta^i_{[n}\delta^j_{m]})=2!\delta^i_{[m}\delta^j_{n]}=\delta^i_m\delta^j_n-\delta^i_n\delta^j_m$. (Or, it can be worked out by expanding all the antisymmetrizers.) It is often quick to use the graphical notation, especially when we have to permute many indices.

A19 By the basic property of antisymmetrizers, = $\frac{1}{4}\Big[$ − + − $\Big]$. The left hand side is zero. Adjoining three upper-right arms by the cross product machine gives = $\frac{1}{4}\Big[$ − + − $\Big]$.

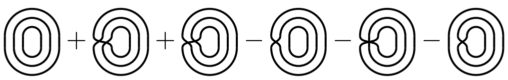
The left hand side is zero as mentioned. Also, note that the cross product machine is already totally antisymmetric so that antisymmetrization of its three indices has no effect on it: = . This leads to



, which is identical to the "bone" identity in **A17**.

A20 + = , but + = $\frac{1}{3}$(+ +) ≠ . Instead, = + $\frac{4}{3}$ + $\frac{4}{3}$ + ; you may want to check it. This is the identity for irreducibly decomposing rank-3 tensors.

A21 Based on the presented argument, we only need to fix the proportionality constant c, where = c . To find the value of c, contract all indices as = c . Then, we have -6 at the left hand side and $c/3!$ times the following at the right hand side.

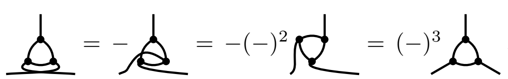
 (69)

$$= 3^3 + 3 + 3 - 3^2 - 3^2 - 3^2 = 6 \qquad (70)$$

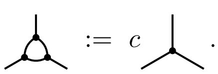
Thus, $-6 = (c/3!) \times 6 = c$.

A22 From rotational invariance and permutation symmetry, The two-terminal expression must be proportional to , so let = c . Connecting the two terminals yields = c ; $-6 = 3c$; $c = -2$.

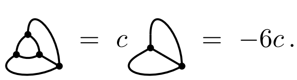
A23 First, we should identify what permutation symmetry has. Swapping two of its indices, we find

 (71)

To flip the whole diagram , we need to flip three cross product machines individually; thus the $(-)^3$ factor. Considering its cyclic symmetry, we conclude that is totally antisymmetric in its three indices. Therefore, it must be proportional to the "unit" totally antisymmetric rank-3 tensor:

 (72)

Attaching another cross-product machine gives

 (73)

Provided that = 6, we can prove that $c = -1$. (Note: the value of is equal to the number of ways to color the edges of the graph with three colors so that distinct colors meet at each vertex.[17] To see why, call the three colors "x," "y," and "z" and remind that there lies ϵ_{ijk} at each vertex with $\epsilon_{xyz} = 1$.)

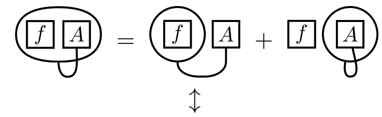
A24 Tadpole diagrams are diagrams such as , , etc. Note that at the tail of tadpole diagrams there always sits a cross product machine.

Consider a tadpole diagram. It can be split into one cross product machine that carries its tail and the rest of it, which has two free terminals so that is proportional to Kronecker delta by Schur's lemma. Thus, any tadpole diagram is proportional to , which is zero: any tadpole diagram vanishes.

This is a consequence of rotational symmetry. An epsilon network must be rotationally invariant because it is composed of cross product machines, which are rotationally invariant. ("Rotationally invariant" means that the "arrowheads" propagate properly along the network; refer to Section I C 3.) However, there are no (nonzero) rotationally invariant tensors that have one terminal: if it existed, it means that it discriminates a particular direction in space. Thus, no nonzero tadpole epsilon networks can exist.

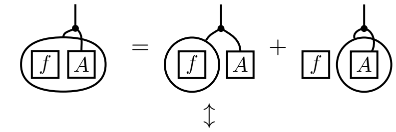
B. Graphical Vector Calculus

A25 (a)

 (74)

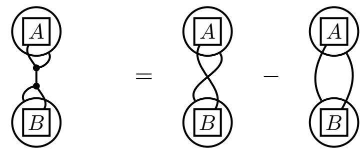
$$\nabla \cdot (f\vec{A}) = \nabla f \cdot \vec{A} + f\,\nabla \cdot \vec{A}$$

(b)

 (75)

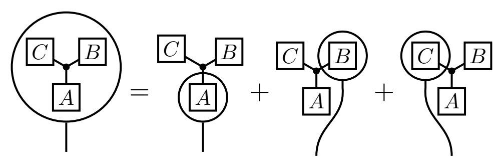
$$\nabla \times (f\vec{A}) = \nabla f \times \vec{A} + f\,\nabla \times \vec{A}$$

A26

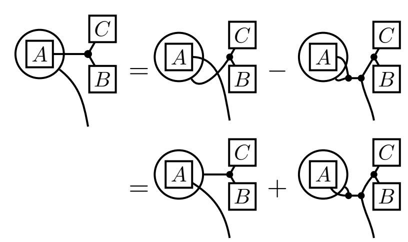
 (76)

$$\frac{\partial A_j}{\partial x_i}\frac{\partial B_m}{\partial x_l}\epsilon_{ijk}\epsilon_{lmk} = \frac{\partial A_j}{\partial x_i}\frac{\partial B_j}{\partial x_i} - \frac{\partial A_j}{\partial x_i}\frac{\partial B_i}{\partial x_j}$$

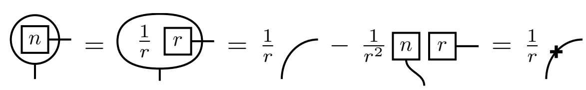
A27 (a)



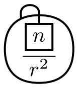
(b) Using the trick,

 (77) (78)

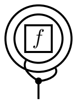
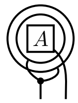
A28

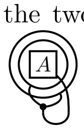
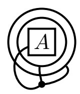
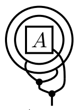


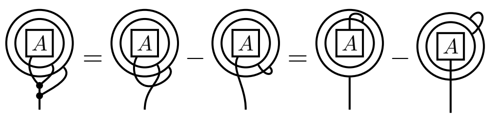
$\vec{n}$ at a position $\vec{r}$ does not change when $\vec{r}$ makes a radial displacement. When $\vec{r}$ is infinitesimally varied along a tangential direction, i.e., the infinitesimal displacement $d\vec{r}$ is perpendicular to $\vec{r}$, $\vec{n}$ rotates by an angle $\frac{1}{r}|d\vec{r}|$ so that it changes by $d\vec{n} = \frac{1}{r}d\vec{r}$. Therefore, $dn_i = \frac{1}{r}P_{ij}\,dx_j$, where P_{ij}, the $\vec{n}$-projector $\delta_{ij} - n_i n_j$, ensures that $\vec{n}$ only react to tangential displacement.

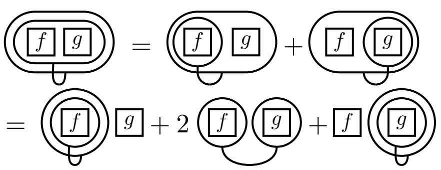
A29  $= 4\pi\delta^{(3)}(\vec{r})$.

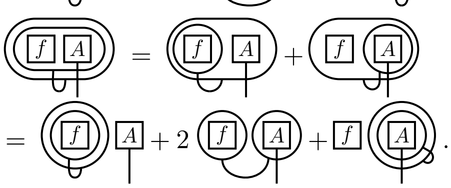
A30 (a) As mentioned in the main article,  $= 0$ holds as an operator identity. Feeding a scalar field $f(\vec{r})$ and a vector field $\vec{A}(\vec{r})$ gives

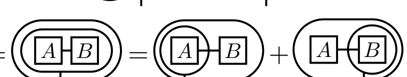
 $= 0$ and  $= 0$, (79)

respectively. The former proves the first identity of the problem. The latter is rarely mentioned in the literature, but if we connect the two ends with Kronecker delta, we have $0 =$  $=$ , which translates into the plaintext notation as $\nabla \cdot (\nabla \times \vec{A}) = 0$. Note that if we use a cross product machine instead of Kronecker delta, we find that  $= 0$, which translates into $\nabla(\nabla \cdot \vec{A}) - \partial_i \nabla(A_i) = 0$.

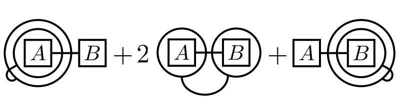
(b) 

A31 (a) 

(b) 

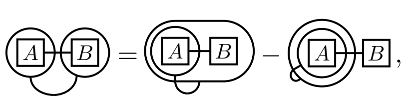
(c) $\nabla^2(\vec{A}\cdot\vec{B}) =$ 

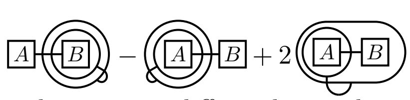
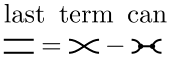
One can proceed in the same way as the former problems to get

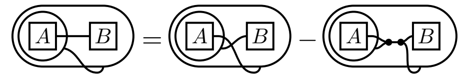
 (80)

In addition, let us try to obtain an expression that is also tractable in the index-free plaintext notation.

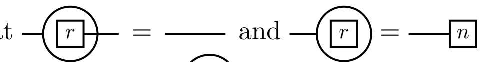
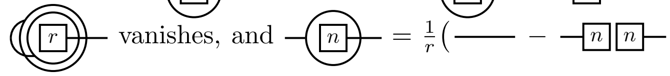
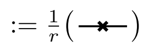
From

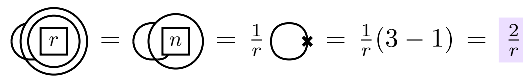
 (81)

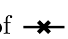
$\nabla^2(\vec{A}\cdot\vec{B}) =$ . The last term can be written differently employing the  trick.

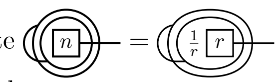
 (82)

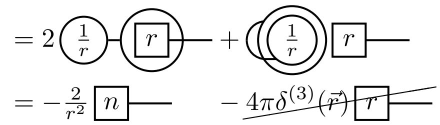
To sum up, $\nabla^2(\vec{A}\cdot\vec{B})$ is equal to $\vec{A}\cdot\nabla^2\vec{B} - \vec{B}\cdot\nabla^2\vec{A}$ plus the double of Eq. (82) which, as a total, result in $\vec{A}\cdot\nabla^2\vec{B} - \vec{B}\cdot\nabla^2\vec{A} + 2\nabla\cdot\Big((\vec{B}\cdot\nabla)\vec{A} - \vec{B}\times(-\nabla\times\vec{A})\Big)$. This proves the equation given by the problem.

A32 Recall that . Also,  .

(a)  $\frac{2}{r}$.

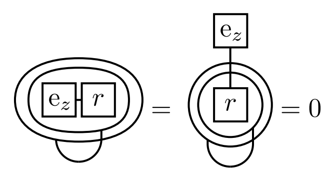
The fact that the trace of  is equal to 2 reflects its dimensionality: it projects vectors to a two-dimensional subspace.

For $\nabla^2\vec{n}$, write  and then apply the Leibniz rule:

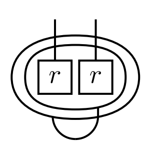
 (83) (84)

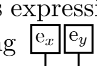
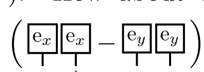
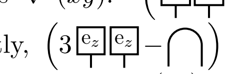
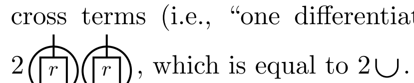
Thus, $-r^2\nabla^2\vec{n} = 2\vec{n}$. This "2" originates from $\ell(\ell+1)$, $\ell = 1$.

(b) $\nabla^2 z = \nabla^2(\vec{e}_z\cdot\vec{r})$, and $\vec{e}_z$, a constant vector, can permeate through the differentiation loop. Therefore, since $\nabla^2\vec{r} = 0$, $\nabla^2 z = 0$.

 (85)

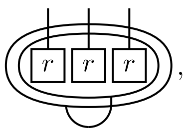
(c) Use the same strategy with the previous problem. First, consider

 (86)

Contracting the two terminals of this expression with  gives $\nabla^2(r^2)$. How about using ? This gives $\nabla^2(xy)$.  gives $\nabla^2(x^2 - y^2)$. Lastly,  gives $\nabla^2(3z^2 - r^2)$. Hence, calculating Eq. (86) will provide all answers for this question. It proceeds by applying the Leibniz rule. Since $\nabla^2\vec{r} = 0$, the only surviving term is the cross terms (i.e., "one differentiation per one $\vec{r}$"), .

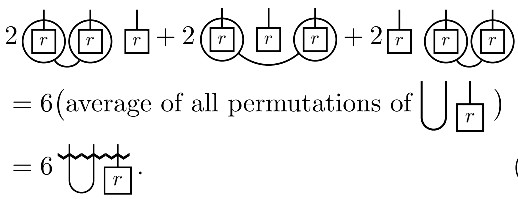
Thus, $\nabla^2(r^2) = 2\bigcirc = 6$. (This "6" originates from $\ell(\ell+1)$, $\ell = 2$.) For the cases of contracting Eq. (86) with $e_x e_y$, $(e_x e_x - e_y e_y)$, and $(3 e_z e_z - \cap)$, you can easily check that they are zero! Therefore, $\nabla^2(3z^2 - r^2), \nabla^2(x^2 - y^2), \nabla^2(xy) = 0$.

(d) In the same way, we are interested in



(87)

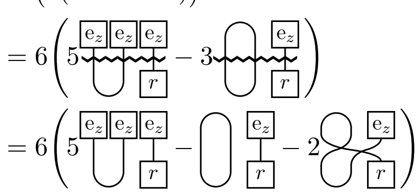
which is equal to



(88)

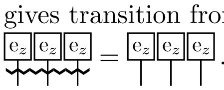
Then, we can obtain $\nabla^2\big(z(5z^2 - 3r^2)\big)$, $\nabla^2\big(x(5z^2 - r^2)\big)$, etc. by plugging in appropriate combination of basis vectors.

$$\nabla^2\big(z(5z^2 - 3r^2)\big) \tag{89}$$

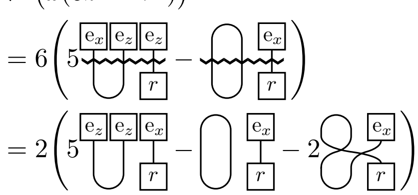


(90) (91)

$$= 6\big(5z - 3z - 2z\big) = 0 \tag{92}$$

Untying symmetrizers into all possible permutations gives transition from Eq. (90) to Eq. (91). Note that . Similarly,

$$\nabla^2\big(x(5z^2 - r^2)\big) \tag{93}$$

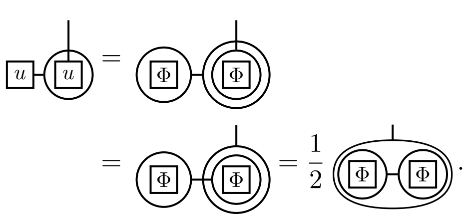


(94) (95)

$$= 2(5x - 3x - 2x) = 0\,. \tag{96}$$

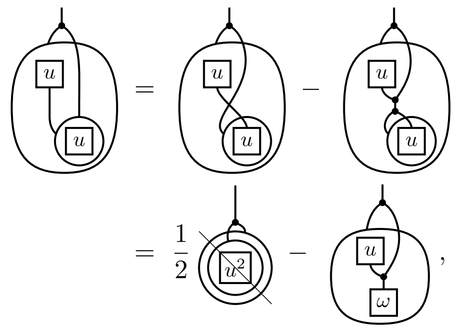
In fact, you can check that $\nabla^2\big(z(x^2 - y^2)\big)$ and $\nabla^2\big(x(x^2 - 3y^2)\big)$ are also zero. Thus, the answers to this problem is 0 .

A33 Letting $\vec{u} = \nabla\Phi$,



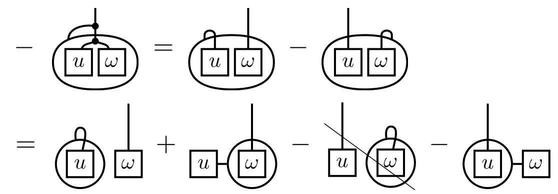
(97) (98)

A34 The proof proceeds in two steps. First,

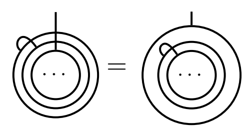


(99) (100)

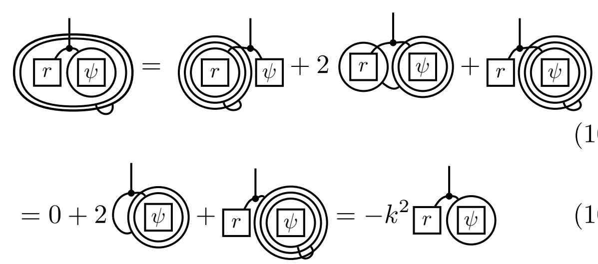
where $\vec{\omega} := \nabla \times \vec{u}$. Then,



(101) (102)

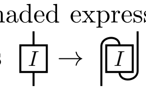
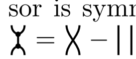
A35 

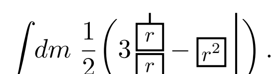
A36 Provided that  $= -k^2 \psi$, the proof proceeds as the following.

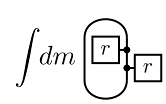
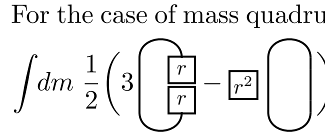
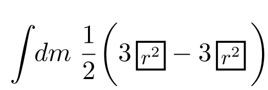


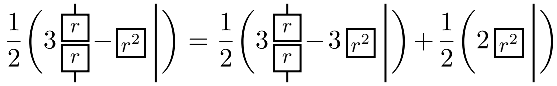
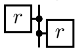
(103) (104)

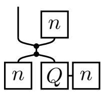
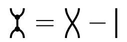
C. Graphical Tensor Calculus

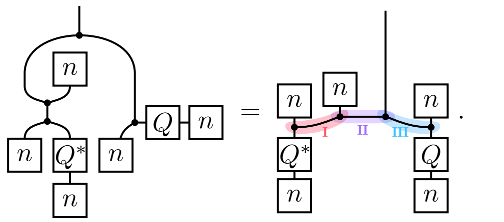
A37 When the two arms of the shaded expression in the middle of Eq. (19) is swapped as  then yanked, it rotates by 180°. As the resulting expression is identical to the expression before swap-then-yanking, the inertia tensor is symmetric. Or, one can expand the expression by  then apply the swap-then-yanking.

A38 

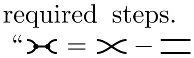
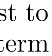
A39 The trace of the inertia tensor reads:  $= \int dm\; 2r^2$ (refer to **P55**a). For the case of mass quadrupole moment, it is traceless:  $=$  $= 0$.

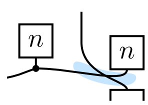
A40  . The first term is equal to $-\frac{3}{2}$  and give $-\frac{3}{2}I_{ij}$ when integrated by $\int dm$. The second term translates into $r^2\delta_{ij}$, and gives $\frac{1}{2}\delta_{ij}I_{kk}$ when integrated, as $\int dm\, 2r^2 = I_{ii}$ shown in the previous problem.

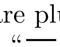
A41 It surely is an "enormous" tensorial expression, but can be wisely reduced further using the graphical notation. First, observe that we can write the bracketed term in Eq. (20) as  . This is easily verified by  and the fact that $\vec{n}$ is a unit vector. Then, $\vec{S}$ is equal to $(\mu_0\omega^6/288\pi^2c^4)r^{-2}$ times the real part of

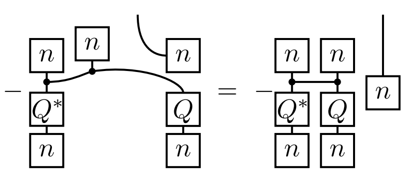


(105)

(Or, if you want, using the plaintext notation, $S_i = (\mu_0\omega^6/288\pi^2c^4)r^{-2}\,\epsilon_{ijl}\epsilon_{jhk}\epsilon_{kab}\epsilon_{lde}n_hn_an_cn_dn_f\,\mathrm{Re}Q^*_{bc}Q_{ef}$.) Here, the asterisk in Q means that it is complex conjugated. One may wonder why we rewrote the bracketed term in Eq. (20) involving two cross products, increasing the net number of cross product machines. In fact, it turns out that it was a wise prescription that effectively reduces the required steps. We have three options for applying the "" identity; among options I, II, and III marked in Eq. (105), what "knot" is the best to cut off first? It is the option III, because the "" term vanishes by $\vec{n}\times\vec{n} = 0$.



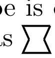
See the two $\vec{n}$'s that are plugged into a cross product machine? Thus, only the "" term survives, and we obtain



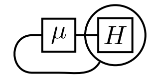
(106)

(One may further expand the  here.) Therefore, the answer is

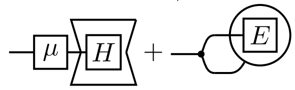
$(\mu_0\omega^6/288\pi^2c^4)r^{-2}$ times the real part of Eq. (106).

A42 It is not necessary to write equations in a completely graphical manner. However, if some want to do so, they can denote $\partial/\partial t$ as a balloon, of which shape is distinguished from that of spatial derivative ∇, such as .

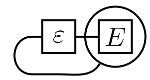
(a) Start with the equations that do not couple with sources. Substituting $\overrightarrow{D} = \overline{\varepsilon}\overrightarrow{E}$ and $\overrightarrow{B} = \overline{\mu}\overrightarrow{H}$,[25]

$\nabla\cdot\overline{\mu}\overrightarrow{H} = 0 \leftrightarrow$  $= 0$, (107)

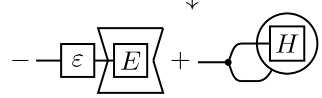
$$\frac{\partial}{\partial t}\overline{\mu}\overrightarrow{H} + \nabla\times\overrightarrow{E} = 0$$
$$\updownarrow$$ (108)

 $= 0$.

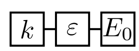
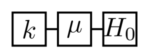
Then, consider the remaining two. Letting $\rho_{\text{free}} = 0$ and $\overrightarrow{J}_{\text{free}} = 0$,

$\nabla\cdot\overline{\varepsilon}\overrightarrow{E} = 0 \leftrightarrow$  $= 0$, (109)

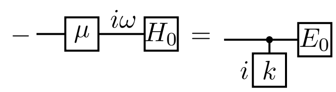
$$-\frac{\partial}{\partial t}\overline{\varepsilon}\overrightarrow{E} + \nabla\times\overrightarrow{H} = 0$$
$$\updownarrow$$ (110)

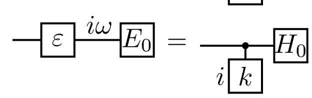
 $= 0$.

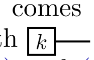
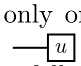
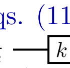
(b) For a plane wave solution $\overrightarrow{E} = \overrightarrow{E_0}e^{-i\omega t + i\overrightarrow{k}\cdot\overrightarrow{x}}$ and $\overrightarrow{H} = \overrightarrow{H_0}e^{-i\omega t + i\overrightarrow{k}\cdot\overrightarrow{x}}$, Eqs. (107) and (109) reads

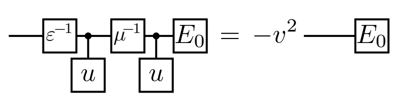
 $= 0$ and  $= 0$, (111)

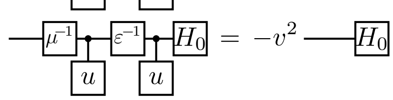
respectively. Next, Eqs. (108) and (110) reads as follows, respectively.

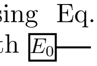
 , (112)

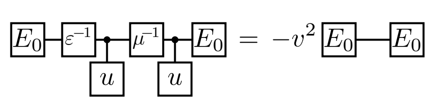
 . (113)

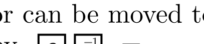
It is evident that Eq. (111) comes from Eqs. (112) and (113) by contracting with . Therefore, one can focus only on Eqs. (112) and (113). Using the definitions  $= \frac{1}{k}$  and $v = \omega/k$, they boil down to the following eigenvalue problems.

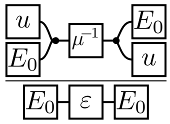
 , (114)

 . (115)

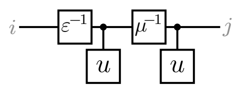
When $\vec{E}_0$ and $\vec{u} := \vec{k}/k$, are known, v^2 can be obtained using Eq. (114); contracting the entire equation with  ,

 . (116)

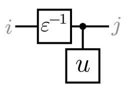
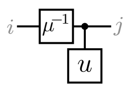
Or, for a more symmetric form, the permittivity tensor can be moved to the right hand side of Eq. (114) (by ) before the contraction:

$$v^2 =$$  (117)

(c) From Eq. (114),

$$W_{ij} =$$  (118)

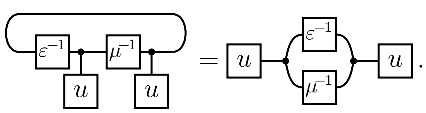
Note: if

$N_{ij} :=$  and $M_{ij} :=$  , (119)

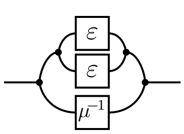
$\overline{W} = \overline{NM}$ (i.e., $W_{ij} = N_{ik}M_{kj}$). $\overline{W}$ gives the eigenvalue problem of $\overrightarrow{E_0}$: $\left(\overline{W} + v^2\overline{1}\right)\overrightarrow{E_0} = 0$. On the other hand, one can find the matrix that gives the eigenvalue problem of $\overrightarrow{H_0} = \overline{MN}$. Provided that $\overline{N}$ is invertible, $\overline{W} = \overline{NM}$ and $\overline{MN}$ are related by similarity transformation so that they have the same eigenvalues.

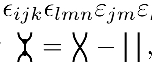
(d) W_{ii}, $\frac{1}{2!}\epsilon_{ijk}\epsilon_{imn}W_{jm}W_{kn}$, and $\frac{1}{3!}\epsilon_{ijk}\epsilon_{lmn}W_{il}W_{jm}W_{kn}$ appear as coefficients in the characteristic equation. We refer interested readers to Peterson's article.[27,28]

i. $\mathrm{tr}\left(\overline{W}\right) = W_{ii} =$

 (120)

This is simple enough; however, if you want to write this in the index-free plaintext notation, $\mathrm{tr}\left(\overline{\varepsilon^{-1}}\,\overline{\star u}\,\overline{\mu^{-1}}\,\overline{\star u}\right)$ can be a choice (refer to **A49**b for the $\star$ notation). Or, try regarding it as a matrix "$\overline{\beta}$" sandwitched with two $\overrightarrow{u}$'s : $\overleftarrow{u}\,\overline{\beta}\,\overrightarrow{u}$. The matrix $\overline{\beta}$ can be massaged as

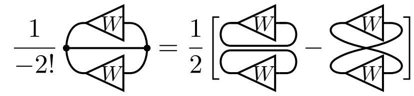
$$\frac{1}{-2!\det\left(\overline{\varepsilon}\right)}$$  , (121)

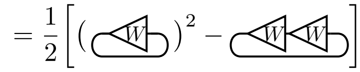
using the expression for the inverse matrix $((\varepsilon^{-1})_{ij} = \frac{1}{\det(\overline{\varepsilon})}\frac{1}{2!}\epsilon_{ijk}\epsilon_{lmn}\varepsilon_{jm}\varepsilon_{kn})$. Untying the  in both sides by , we have

$$\frac{1}{\det\left(\overline{\varepsilon}\right)}\left[\overline{\varepsilon}\,\overline{\mu^{-1}}\,\overline{\varepsilon} - \overline{\varepsilon}\ \mathrm{tr}\left(\overline{\mu^{-1}}\,\overline{\varepsilon}\right)\right], \quad (122)$$

which appears in Taouk's work.[8]

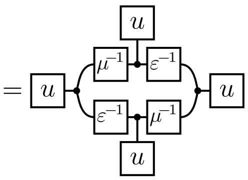
ii. $\left(\mathrm{adj}(W)\right)_{il} = \frac{1}{2!}\epsilon_{ijk}\epsilon_{lmn}W_{jm}W_{kn}$ is called the adjugate matrix of $\overline{W}$. We want to calculate its trace, $\frac{1}{2!}\epsilon_{ijk}\epsilon_{imn}W_{jm}W_{kn}$. In the graphical notation,

 (123)

 (124)

$$= \frac{1}{2}\left[\left(\mathrm{tr}\left(\overline{W}\right)\right)^2 - \mathrm{tr}\left(\overline{W}^2\right)\right]. \quad (125)$$

Writing $\mathrm{tr}\left(\overline{W}^2\right)$ in terms of $\overrightarrow{u}$ and the permittivity and permeability matrices is straightforward.

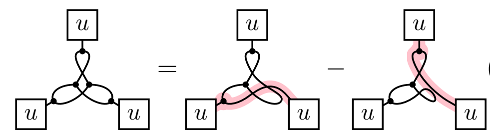
$\mathrm{tr}\left(\overline{W}^2\right) =$  (126)

iii. $\det\left(\overline{W}\right) = \frac{1}{3!}\epsilon_{ijk}\epsilon_{lmn}W_{il}W_{jm}W_{kn}$.

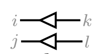
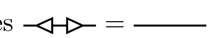
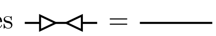
As $\overline{W} = \overline{\varepsilon^{-1}}\,\overline{\star u}\,\overline{\mu^{-1}}\,\overline{\star u}$,

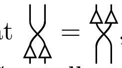
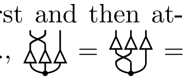
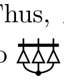
$$\det\left(\overline{W}\right) = \left(\det\left(\overline{\star u}\right)\right)^2 / \det\left(\overline{\varepsilon\mu}\right). \quad (127)$$

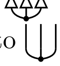
However, $\det\left(\overline{\star u}\right) = 0$, so $\det\left(\overline{W}\right) = 0$. This is because the map $\overrightarrow{V} \mapsto \overline{\star u}\overrightarrow{V}$ is non-invertible, as $\overline{\star u}\,\overrightarrow{u} = \overrightarrow{u} \times \overrightarrow{u} = 0$. Or, by explicit calculation,

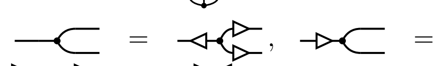
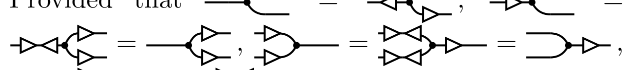
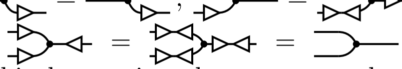
 (128)

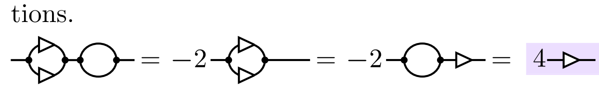
$$= 0 - 0 = 0. \quad (129)$$

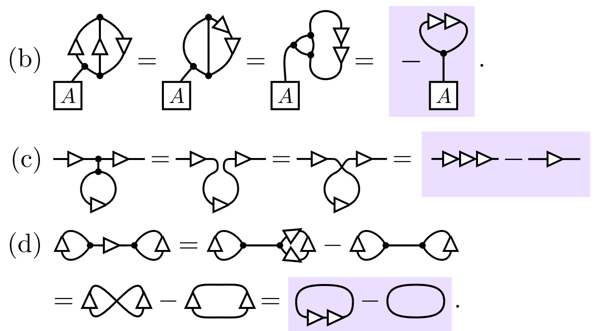
A43 $R_{ik}R_{jl} =$  . Joining k and l gives  . Joining i and j gives  . These two correspond to $R_{ik}R_{jk} = \delta_{ij}$ and $R_{jk}R_{jl} = \delta_{kl}$, respectively.

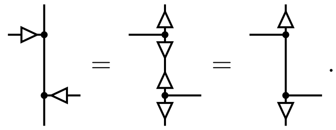
A44 (a) Observe that , if you "slide" the arrowheads upwards. Generally, attaching identical rank-2 tensors and then permuting the indices gives the same result with permuting the indices first and then attaching identical rank-2 tensors: e.g.,  . Any two indices are antisymmetric in the same way. Thus,  is totally antisymmetric so that is equal to .

(b) By Eq. (3), . By the definition of determinant, this is equal to .

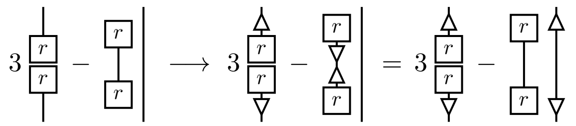
(c) Provided that , , and . The given graphical equation demonstrates these four situations.

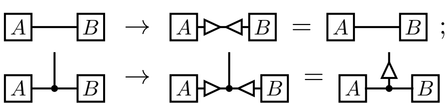
A45 (a)  .

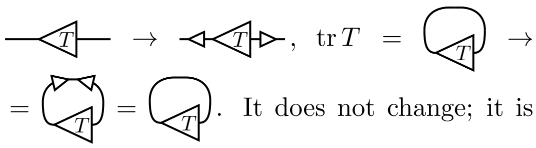


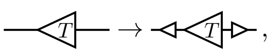
A46 

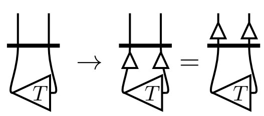
A47 Provided that the integration measure is invariant, the following confirms that the mass quadrupole moment is a tensor:



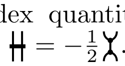
A48 

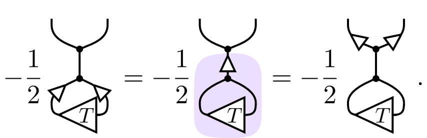
A49 (a) When  . It does not change; it is a scalar.

(b) When ,

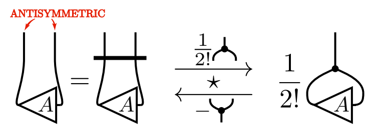
 (130)

(permuting indices and dressing indices with the same matrices commute: see **A44**a). It dresses outward arrowheads at each of its two indices; it is a rank-2 tensor. Since it is antisymmetric ($T_{[ij]} = -T_{[ji]}$), it has only $\binom{3}{2} = \frac{3!}{2!1!} = 3$ independent components (number of ordered pairs (i, j) such that $i < j$ and $i, j \in \{1, 2, 3\}$): $T_{[23]}$, $T_{[31]}$, and $T_{[12]}$.

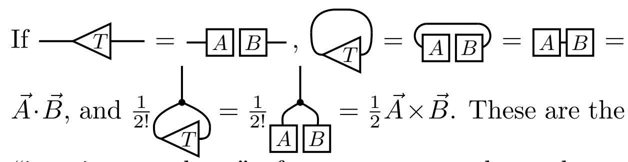
Meanwhile, a vector also has $\binom{3}{1} = 3$ independent components. This suggests that an antisymmetric rank-2 tensor can be reformatted into a vector, preserving the contained information. This can be achieved by employing the epsilon tensor, because it is antisymmetric in its two indices and has three indices so that can convert a two-index quantity to a one-index quantity and vice versa. Use the substitution . Then, Eq. (130) reads:

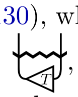
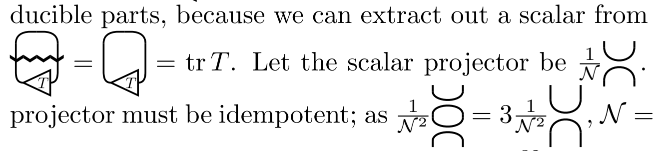
 (131)

Note that although the whole expression is a rank-2 tensor, the purple-shaded area transforms like a rank-1 tensor, i.e., a vector. Therefore, an antisymmetric rank-2 tensor, say, A_{ij}, can be encoded in a vector $(\star A)_i := \frac{1}{2!}\epsilon_{ijk}A_{jk}$ without loss of information so that it can be inverted as $A_{ij} = \epsilon_{ijk}(\star A)_k$ (the normalization factors $\frac{1}{2!}$ and $\frac{1}{1!}$ are conventional). In other words, we can always convert a bivector (a directed area) into a vector (a directed line) by "the right hand" (ϵ_{ijk} or $\star$) and vice versa.

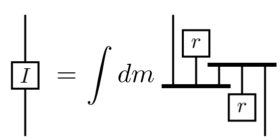


(132)

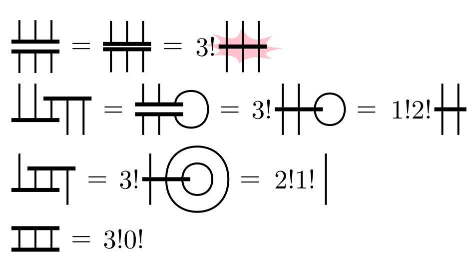
(c) If  $= \frac{1}{2}\vec{A}\times\vec{B}$. These are the "invariant products" of two vectors: scalar and vector products. (Therefore, the invariant products are in general irreducible parts of direct product of two objects. Then, we can also think about the "symmetric traceless product" of two vectors, say, $\vec{A}\cap\vec{B}$, defined as $(\vec{A}\cap\vec{B})_{ij} := A_iB_j - \frac{1}{3}\delta_{ij}A_kB_k$. $\nabla\cap\vec{u}(\vec{r})$ will be related to the "shear"[29] of a vector field $\vec{u}(\vec{r})$.[30])

A50 First, a rank-2 tensor T_{ij} can be divided into its antisymmetric part and the symmetric part. The former gives Eq. (130), which has three independent components. The latter, , can be further decomposed into irreducible parts, because we can extract out a scalar from it:  $= \operatorname{tr} T$. Let the scalar projector be $\frac{1}{\mathcal{N}}$ ... A projector must be idempotent; as ... $\mathcal{N} = 3$. Hence, we have the decomposition Eq. (26).[32]

A51 Our definition of the inertia tensor is a machine that has one input for an angular velocity vector and one output for giving the corresponding angular momentum vector (Eq. (19)). Thus, it is a $\binom{1}{1}$-tensor. In the graphical notation that distinguishes contravariant and covariant indices,

 (133)

A52



(134)

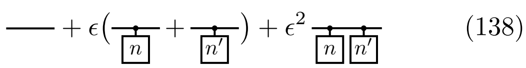
(135)

(136)

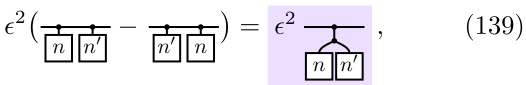
(137)

A53 (a) A geometric reasoning for a moment reveals that a vector $\vec{r}$ is transformed into $\vec{r}' = \vec{r} + \epsilon\vec{n}\times\vec{r}$ under an infinitesimal rotation of angle ϵ with respect to axis $\vec{n}$. From $R_{ij}x_j = x'_i = x_i + \epsilon\,\epsilon_{ijk}n_jx_k + \mathcal{O}(\epsilon^2)$, we obtain $R_{ij} = \delta_{ij} + \epsilon\,\epsilon_{ijk}n_j$, which translates into $= \;+ \epsilon \;+ \mathcal{O}(\epsilon^2)$.

(b) Substituting the result obtained in the previous problem, is equal to



up to $\mathcal{O}(\epsilon^2)$. We can see that infinitesimal rotation matrices are additive up to $\mathcal{O}(\epsilon^1)$. Then, $-$ is order of ϵ^2:

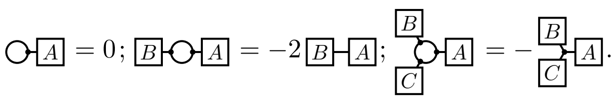


where we used the Jacobi identity, Eq. (63), at the last step.

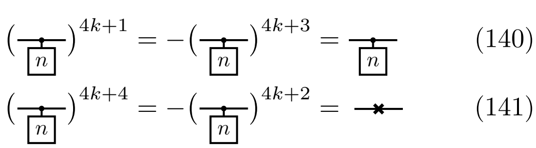
A54 (a) Since it is an antisymmetric rank-2 tensor, it lives in the rank-2, spin-1 (antisymmetric) representation: generators of SO(3), the elements of so(3), are real antisymmetric matrices.

(b) From completeness, we know that $+$ $+$ $=$. Thus, the given equation is equal to $-$ $= 2$; the quadratic Casimir is 2.

A55 (a)

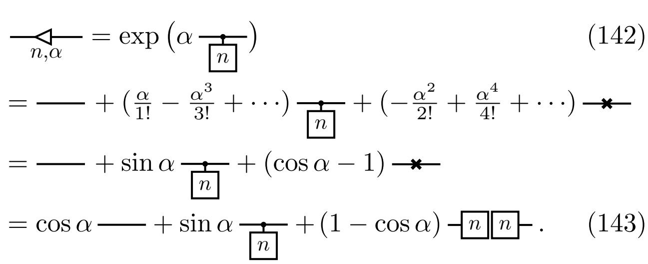


(b) $(\;)^0 =$, and $(\;)^1 =$. $(\;)^2 = -$, as calculated in **P8**. Now, recall that , a shorthand for $-$, selects components that are orthogonal to $\vec{n}$; therefore, $(\;)^3 = - \;= -$. This is proportional to $(\;)^1$: the sequence is periodic. Hence,



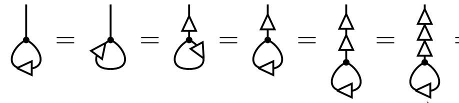
for any integer $k \geq 0$, and $(\;)^0 =$.

A56 From the results of the previous problem,



A57 (a) tr $(\;) = \cos\alpha$ tr $(\;) + \sin\alpha$ tr $(\;) + (1 - \cos\alpha)$ tr $(\;) = 3\cos\alpha + 0 + (1-\cos\alpha) = 1 + 2\cos\alpha$.

(b)

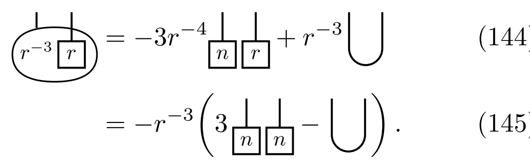


(the label "n, α" omitted to avoid clutter); this little one-dimensional creature living on a golden braided knot proliferates eternally![12] This means that is a distinguished vector: an eigenvector with eigenvalue one of the rotation , i.e. it remains the same after being rotated. Therefore, we conclude that it is parallel to the rotation axis.

Using Rodrigues' formula, we can explicitly calculate it in terms of $\vec{n}$ and α: $= \frac{-1}{2\sin\alpha}$ (or, $= \; - \; = 3 - \text{tr}\,(\;) = 3 - (1 + 2\cos 2\alpha) = 4\sin^2\alpha$.) This provides a formula for finding the corresponding rotation axis when a rotation matrix is given.

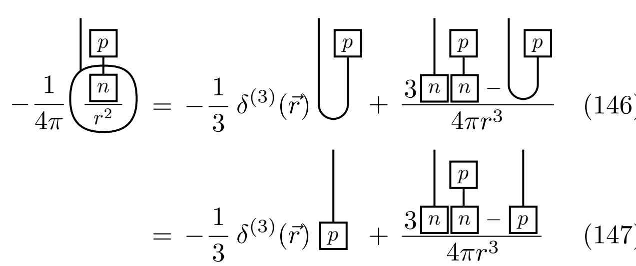
(c) Since tr $(\;) = 1 + 2\cos\alpha$, the spin-2 part of is equal to $\frac{1}{2}(\; + \;) - (1 + 2\cos\alpha)/3 \; = (1 - \cos\alpha)(\; - \;)$.

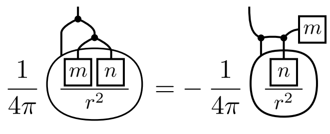
A58 At $r > 0$,



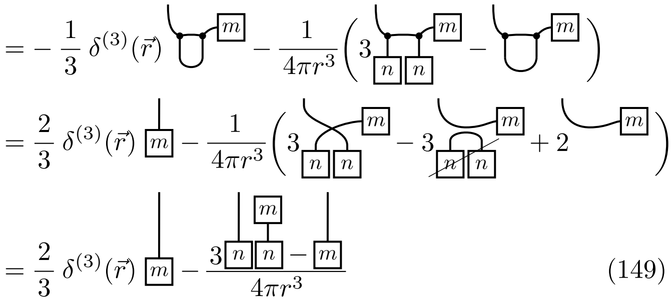
This confirms Eq. (29) except for the delta function term. (Note that Eq. (145) is traceless and symmetric; it is spin-2. There is no spin-1 part because $\nabla \times (\vec{n}/r^2) = 0$. Recall that $\partial_{[i}(n_{j]}/r^2)$ and $\nabla \times (\vec{n}/r^2)$ are compatible, as explained in **A49**b.) However, to examine whether there sits a delta function at the origin, we have to investigate the behavior of $\vec{n}/r^2$ at $r \to 0$, while it is well-known that $\nabla \cdot (\vec{n}/r^2)$, i.e., trace of $\partial_i(n_j/r^2)$, is $4\pi\delta^{(3)}(\vec{r})$. Assume that the singular term of $\partial_i(n_j/r^2)$ is $c_{ij}\delta^{(3)}(\vec{r})$, where c_{ij} is a constant tensor. Then, c_{ij} should not have a particular preferred direction, because $\vec{n}/r^2$ is an isotropic vector field. Its components must be unchanged under rotation, i.e., it must be a scalar. Thus, c_{ij} only has the trace-only part, $\frac{1}{3}\delta_{ij}c_{kk}$. $\nabla \cdot (\vec{n}/r^2) = 4\pi\delta^{(3)}(\vec{r})$ confirms that $c_{kk} = 4\pi$. Therefore, the delta function term is $\frac{4\pi}{3}\delta_{ij}\delta^{(3)}(\vec{r})$. To sum up, $\partial_i(n_j/r^2)$ has spin-0 and spin-2 parts but no spin-1 part, and the two parts are the delta function term and Eq. (145), respectively.

A59 Since $\vec{p}$ and $\vec{m}$ are constant vectors, they can be freely overpass the differentiation loop. Proofs of Eq. (27) and Eq. (28) proceed as follows.



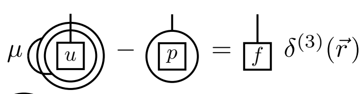
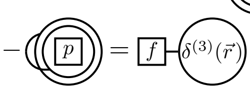
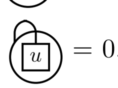
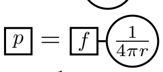
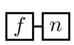
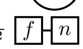


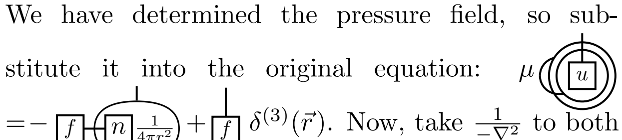
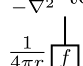
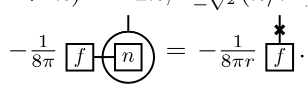
(148)

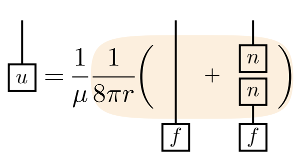


(149)

A60 Note that, for a scalar field $\Phi(\vec{r})$, if, $-\nabla^2\Phi(\vec{r}) = \delta^{(3)}(\vec{r})$ and $\Phi(\vec{r}) \to 0$ as $r \to \infty$, then $\Phi(\vec{r}) = \frac{1}{4\pi r}$. Let us denote this as "$\Phi(\vec{r}) = \frac{1}{-\nabla^2}\delta^{(3)}(\vec{r}) = \frac{1}{4\pi r}$" in shorthand. If the boundary condition is different, we formally write "$\Phi(\vec{r}) = \frac{1}{-\nabla^2}\delta^{(3)}(\vec{r}) + \frac{1}{-\nabla^2}0$," where $\frac{1}{-\nabla^2}0$ means an element in $\ker[-\nabla^2]$ (i.e., a term that vanishes when being subjected to $-\nabla^2$) that is responsible for matching the boundary conditions.

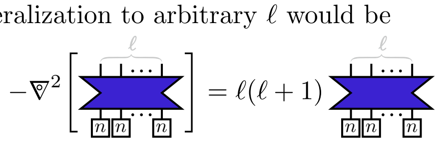
(a) Taking divergence to  gives , since  $= 0$. Taking $\frac{1}{-\nabla^2}$ to this equation gives  $+ \frac{1}{-\nabla^2}0 =$ $-\frac{1}{4\pi r^2}$  $+ \frac{1}{-\nabla^2}0$. Since $-\frac{1}{4\pi r^2}$  $\to 0$ when $r \to \infty$, the homogeneous solution $\frac{1}{-\nabla^2}0$ should be p_0 provided that the given boundary condition holds.

(b) We have determined the pressure field, so substitute it into the original equation:  $\delta^{(3)}(\vec{r})$. Now, take $\frac{1}{-\nabla^2}$ to both sides. The delta-function term becomes . How about the first term in the right hand side? Note that $-r^2\nabla^2\vec{n} (= -\nabla\!\!\!\!\nabla^2\vec{n}) = 2\vec{n}$; $\frac{1}{-\nabla^2}(\vec{n}/r^2) = \frac{1}{2}\vec{n}$. Thus, it becomes . Adding the two, we obtain the following. (The tensor in the shaded area is called the Oseen tensor.)



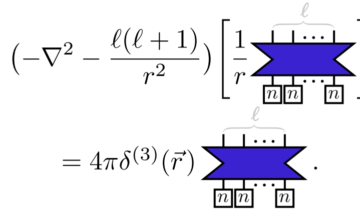
(150)

A side note: we found $-\nabla\!\!\!\!\nabla^2\vec{n} = 2\vec{n}$ fruitful. Its generalization to arbitrary ℓ would be

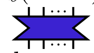


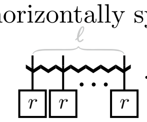
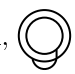
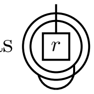
(151)

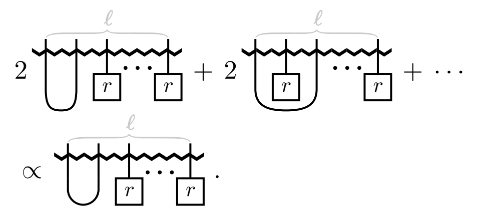
so that



(152)

A61 Since $\mathrm{P}_\ell(\vec{n}' \cdot \vec{n}) = \mathrm{P}_\ell(\vec{n} \cdot \vec{n}')$ for arbitrary unit vectors $\vec{n}$ and $\vec{n}'$,  should be the same if the upper terminals and the lower terminals are swapped. The *self-explanatory* design of this is to choose a horizontally symmetric shape.

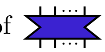
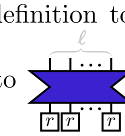
A62 Consider ℓ symmetrized $\vec{r}$'s: . Taking Laplacian, , to this, as  $= 0$, we get

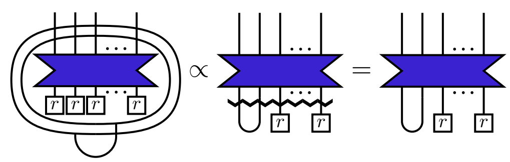


(153)

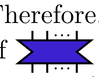
(154)

Refer to the calculations in **A32**. Note that the equality from Eq. (153) to Eq. (154) holds because the ℓ lines are symmetrized so that all terms in Eq. (153) are identical. (The proportionality constant will be $2\binom{\ell}{2}$, since the number of ways to connect two lines out of ℓ is $\binom{\ell}{2}$.)

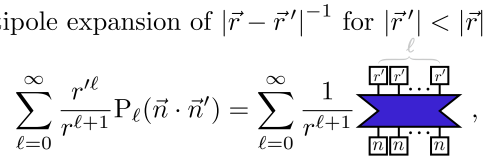
Note that the ℓ ends of  are by definition totally symmetric. Therefore, taking Laplacian to  will give the following.



(155)

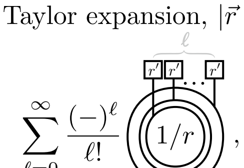
Therefore, connecting the first and second lower indices of  gives zero; so do any of its two lower indices; so do any of its upper indices. (But not for one upper index and one lower index.) Thus, it is "traceless."

A63 Multipole expansion of $|\vec{r} - \vec{r}'|^{-1}$ for $|\vec{r}'| < |\vec{r}|$ gives



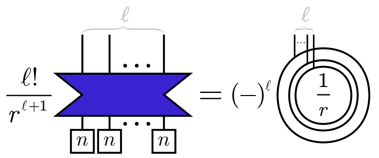
(156)

where $r := |\vec{r}|$, $r' := |\vec{r}'|$, $\vec{r} := r\vec{n}$, and $\vec{r}' := r'\vec{n}'$. Meanwhile, from Taylor expansion, $|\vec{r} - \vec{r}'|^{-1}$ is equal to



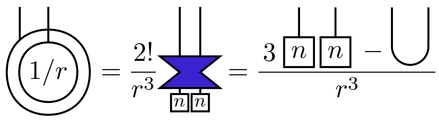
(157)

i.e., shifting $|\vec{r}|^{-1} = 1/r$ by $\vec{r} \to \vec{r} - \vec{r}'$ (thus the factor $(-)^\ell$). At first glance, Eq. (158) may be concluded by comparing Eqs. (156) and (157) according to the order of $\vec{r}'$:



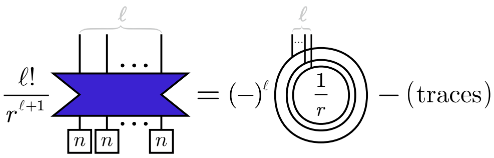
(158)

However, there is a caveat here: this argument only works for $\vec{r} \neq 0$. There can sit singularities at $r = 0$. To see this, consider the case $\ell = 2$. We have



(159)

which is Eq. (29) without the delta function term. In fact, Eq. (158) is an "illegal" tensor equation, because the left hand side is totally symmetric and traceless (as we have shown in **P62**), while the right hand side is totally symmetric but not traceless. Therefore, we need "−(traces)."



(160)

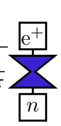
While $\nabla^2(1/r) = -4\pi\delta^{(3)}(\vec{r})$, the "−(traces)" contains $(\ell-2)^{\text{th}}$-order, $(\ell-4)^{\text{th}}$-order, $\cdots$ derivatives of $4\pi\delta^{(3)}(\vec{r})$. For example, for $\ell = 2$, the trace term we should subtract from (1/r) is $\frac{1}{3}$(1/r), which is equal to $-\frac{4\pi}{3}\delta^{(3)}(\vec{r})$.

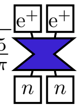
A64 (a) For $\ell = 2$, $= \frac{3}{2}$ $- \frac{1}{2}$ is traceless, as $\frac{3}{2}$ $- \frac{1}{2}$ $= \frac{3}{2}$ $- \frac{1}{2}3$ $= 0$.

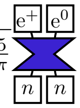
For $\ell = 3$, we have $= \frac{5}{2}$ $- \frac{3}{2}$ and $\frac{5}{2}$ $- \frac{3}{2}$ $= \frac{5}{2}$ $- \frac{3}{2}\frac{5}{3}$ $= 0$, as 3 $=$ $+$ $+$ $= 3$ $+$ $+$ $= 5$.

(b) Let $= a_4$ $+ a_2$ $+ a_0$. (Note that a_3 and a_1 terms cannot appear.) Then, tracelessness implies a_4 $+ a_2$ $+ a_0$ $= 0$. Doodling diagrams for a moment, one can see that $\binom{4}{2}$ $= 5$ $+ 2$ $+$. Using this, one obtains $(a_4 + \frac{7}{6}a_2)$ $+ (\frac{1}{6}a_2 + \frac{10}{6}a_0)$ $= 0$. Therefore, $a_4 : a_2 : a_0 = 35 : -30 : 3$. Since $P_4(\vec{n}\cdot\vec{n}) = 1$, $a_4 + a_2 + a_0 = 1$. In conclusion, $a_4 = \frac{35}{8}$, $a_2 = -\frac{30}{8}$, and $a_0 = \frac{3}{8}$.

A65 Recall that $=$ and $= \frac{3}{2}$ $- \frac{1}{2}$. Also, note that $\vec{e}^+ \cdot \vec{e}^+ = 0$ and $\vec{e}^- \cdot \vec{e}^- = 0$.

- $\sqrt{\frac{3}{4\pi}}$  $= \sqrt{\frac{3}{4\pi}}\vec{e}^+ \cdot \vec{n} = \sqrt{\frac{3}{4\pi}}(-\frac{x+iy}{\sqrt{2}r})$
  $= -\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\phi} = Y_1^{+1}(\theta,\phi)$;

- $\frac{2}{3}\sqrt{\frac{15}{8\pi}}$  $= \frac{2}{3}\sqrt{\frac{15}{8\pi}}\big(\frac{3}{2}(\vec{e}^+ \cdot \vec{n})^2 - \frac{1}{2}(\vec{e}^+ \cdot \vec{e}^+)(\vec{n}\cdot\vec{n})\big)$
  $= \sqrt{\frac{15}{8\pi}}(-\frac{x+iy}{\sqrt{2}r})^2 = \sqrt{\frac{15}{32\pi}} \sin^2\theta\, e^{2i\phi} = Y_2^{+2}(\theta,\phi)$;

- $\frac{2}{3}\sqrt{\frac{15}{4\pi}}$ 
  $= \frac{2}{3}\sqrt{\frac{15}{4\pi}}\big(\frac{3}{2}(\vec{e}^+ \cdot \vec{n})(\vec{e}^0 \cdot \vec{n}) - \frac{1}{2}(\vec{e}^+ \cdot \vec{e}^0)(\vec{n}\cdot\vec{n})\big)$
  $= \sqrt{\frac{15}{4\pi}}(-\frac{x+iy}{\sqrt{2}r})\frac{z}{r}$
  $= -\sqrt{\frac{15}{8\pi}} \sin\theta\cos\theta\, e^{i\phi} = Y_2^{+1}(\theta,\phi)$.

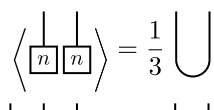
One can obtain further spherical harmonics in a similar manner. Note that the normalization constants — $\sqrt{\frac{3}{4\pi}}$, $\frac{2}{3}\sqrt{\frac{15}{8\pi}}$, $\frac{2}{3}\sqrt{\frac{15}{4\pi}}$, and all that — can be found algebraically (rather than doing integrals in a typical manner) by the following identity, where $\int d^2\Omega = \int \sin\theta d\theta\, d\phi$ is the solid angle integral.

$$\langle n_{i_1} n_{i_2} \cdots n_{i_{2\ell}} \rangle := \frac{1}{4\pi}\int d^2\Omega\; n_{i_1} n_{i_2} \cdots n_{i_{2\ell}} \tag{161}$$
$$= \frac{1}{2\ell+1}\delta_{(i_1 i_2} \cdots \delta_{i_{2\ell-1} i_{2\ell})} \tag{162}$$

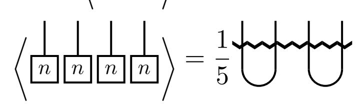
That is, the spherical average of $n_{i_1} n_{i_2} \cdots n_{i_\ell}$ is equal to the average of "all possible contractions" of indices i_1 to $i_{2\ell}$ by Kronecker delta divided by $2\ell+1$. For example,

$$\langle n_i n_j \rangle = \frac{1}{3}\delta_{ij}, \tag{163}$$
$$\langle n_i n_j n_k n_l \rangle = \frac{1}{5}\left(\frac{1}{3}(\delta_{ij}\delta_{kl} + \delta_{li}\delta_{jk} + \delta_{ik}\delta_{jl})\right). \tag{164}$$

Eqs. (163) and (164) show up in standard textbooks.[33,34] In the graphical notation,

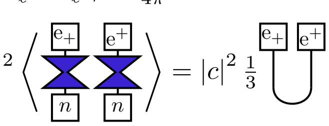


(165)



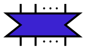
(166)

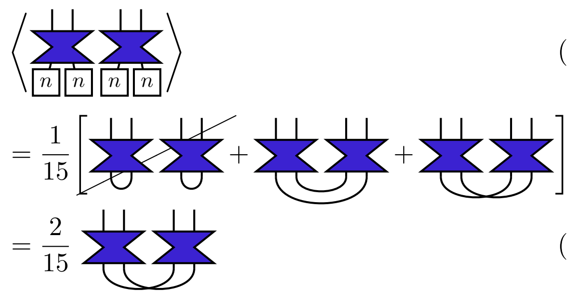
Note that $= \frac{1}{3}\Big($ $+$ $+$ $\Big)$. Then, demanding $\langle Y_\ell^{m*} Y_\ell^m \rangle = \frac{1}{4\pi}$,

- $\frac{1}{4\pi} = |c|^2$  $= |c|^2\,\frac{1}{3}$ $= |c|^2\,\frac{1}{3}$
  $\Longrightarrow c := \sqrt{\frac{3}{4\pi}}$.

Here, $\vec{e}_+ = -\frac{1}{\sqrt{2}}(\vec{e}_x - i\vec{e}_y)$ is a dual vector; numerically, $\vec{e}_+ := (\vec{e}^+)^*$. Similarly, $\vec{e}_0 := (\vec{e}^0)^* = \vec{e}_z$ and $\vec{e}_- := (\vec{e}^-)^* = \frac{1}{\sqrt{2}}(\vec{e}_x + i\vec{e}_y)$. These dual basis vectors satisfy $\vec{e}^{m'} \cdot \vec{e}_m = \delta_m^{m'}$ $(m, m' = +, 0, -)$.

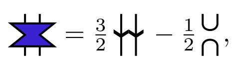
For Y_2^{+2} and Y_2^{+1}, note that the following holds because

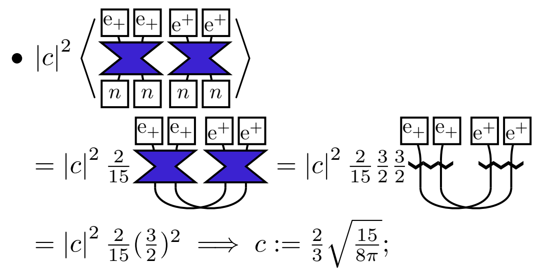
 is traceless and symmetric.

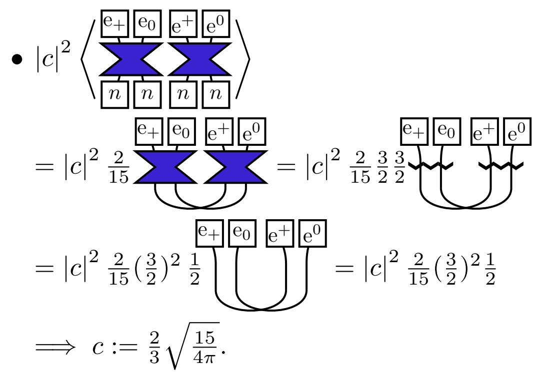


(167)

(168)

Then, using Eq. (168) with ,

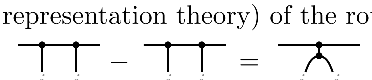


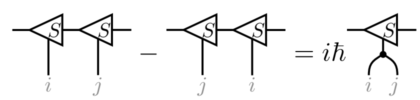


Note that the identity Eq. (162) enables us to do spherical integrals in a coordinate-free way. It turns out to be useful in other calculations too. For example, $\langle\cos^4\theta\sin^2\theta\sin^2\phi\rangle = \langle(\vec{e}_z\cdot\vec{n})^4(\vec{e}_y\cdot\vec{n})^2\rangle = \frac{1}{7}($average of all possible contractions of $\vec{e}_z,\vec{e}_z,\vec{e}_z,\vec{e}_z,\vec{e}_y,\vec{e}_y) = \frac{1}{7}\left[\frac{1}{6!}\cdot\binom{3}{1}2!4!\right] = \frac{1}{35}$, counting the number of ways for the grouping $((\vec{e}_y,\vec{e}_y),(\vec{e}_z,\vec{e}_z),(\vec{e}_z,\vec{e}_z))$; $\langle\cos^2\theta\sin^4\theta\sin^2\phi\cos^2\phi\rangle = \langle(\vec{e}_z\cdot\vec{n})^2(\vec{e}_x\cdot\vec{n})^2(\vec{e}_y\cdot\vec{n})^2\rangle = \frac{1}{7}\left[\frac{1}{6!}\cdot 3!\,2!^3\right] = \frac{1}{105}$, counting the number of ways for the grouping $((\vec{e}_x,\vec{e}_x),(\vec{e}_y,\vec{e}_y),(\vec{e}_z,\vec{e}_z))$.

A66 By "operator," we mean that $i\hbar$ [diagram] maps a vector into another vector through its terminals drawn horizontally. It is one of the possible realizations of the spin operator: spin operator for spin-1 states (for example, for spin-1/2 states, we cannot use $i\hbar$ [diagram]).

The sudden appearance of $\hbar$ here may evoke an impression that we are making a stark transition to the quantum world, but in fact it is not a necessary one; one could have defined [diagram], not $i\hbar$ [diagram], as a spin operator. Usually, physicists prefer to carry $i\hbar$ to let the generators be Hermitian (observables). In a sense, we are not really introducing quantum mechanics but doing representation "practice" (cf. representation theory) of the rotation group.

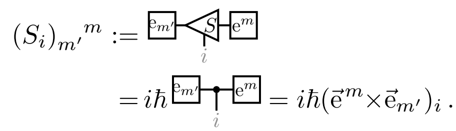
(a)  is the Jacobi identity that we have discussed in **P6**a, **P11**, and **P16**. Multiplying $(i\hbar)^2$ gives the proposed equation. Then, what is its interpretation? Let $i\hbar$ [diagram] := [diagram]. Then, we have



(169)

This is the familiar "$[\hat{S}_i,\hat{S}_j] = i\hbar\epsilon_{ijk}\hat{S}_k$!"

One can explicitly assure that [diagram] is the spin operator for spin-1 states by calculating its components ("matrix elements"). Sandwiching with $\vec{e}_{m'}$ and $\vec{e}^m$,



(170)

$= i\hbar(\vec{e}^m\times\vec{e}_{m'})_i\,.$ (171)

Displaying these numbers in a matrix format, $m' = +,0,-$ labelling rows and $m = +,0,-$ labelling columns, we get the following.

$$\left[(S_3)_{m'}{}^m\right] = \hbar \begin{bmatrix} 1 & & \\ & 0 & \\ & & -1 \end{bmatrix} \tag{172}$$

$$\left[(S_1)_{m'}{}^m\right] = \frac{\hbar}{\sqrt{2}} \begin{bmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{bmatrix} \tag{173}$$

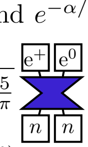
$$\left[(S_2)_{m'}{}^m\right] = \frac{\hbar}{\sqrt{2}} \begin{bmatrix} & -i & \\ i & & -i \\ & i & \end{bmatrix} \tag{174}$$

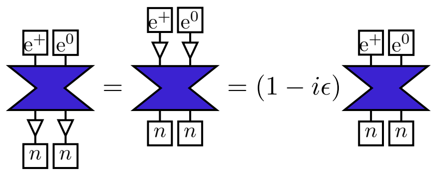
(rows and columns labelled $+, 0, -$)

(Notice: zeros sitting in the blank spaces, omitted to avoid clutter.) These matrices should be familiar from undergraduate quantum mechanics.

(b) Explicit calculation with $\vec{e}^+ = -\frac{1}{\sqrt{2}}(\vec{e}_x + i\vec{e}_y)$, $\vec{e}^0 = \vec{e}_z$, and $\vec{e}^- = \frac{1}{\sqrt{2}}(\vec{e}_x - i\vec{e}_y)$ reveals that $i\hbar\vec{e}_z\times\vec{e}^+ = +\hbar\vec{e}^+$, $i\hbar\vec{e}_z\times\vec{e}^0 = 0\hbar\vec{e}^0$, and $i\hbar\vec{e}_z\times\vec{e}^- = -\hbar\vec{e}^-$; $\vec{e}^+$, $\vec{e}^0$, and $\vec{e}^-$ are the spin-1 eigenstates $|1,1\rangle$, $|1,0\rangle$, and $|1,-1\rangle$, respectively. (Note that their eigenvalues with respect to the Casimir operator $\hat{S}^2 = (i\hbar)^2$ [diagram] $= 2\hbar^2$ [diagram] (refer to **P54**b) are $2\hbar^2$.)

"$i\hbar\vec{e}_z\times$"$=$ $i\hbar$ [diagram] is the generator of z-axis rotation, while the rotation matrix corresponding to z-axis rotation of rotation angle α is given by [diagram] $= \exp(\alpha$ [diagram]$) = \exp(\frac{\alpha}{i\hbar}i\hbar$ [diagram]$)$. $\vec{e}^+$, $\vec{e}^0$, and $\vec{e}^-$ are the eigenvectors of the generator of z-axis rotation and so are the eigenvectors of z-axis rotation. They gain phase factors $e^{\alpha/i}$, 1, and $e^{-\alpha/i}$ under [diagram], respectively.

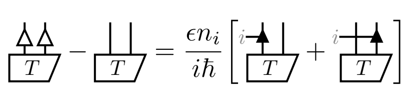
(c) From **P65**, we know that $\mathrm{Y}_2^{+1}(\theta,\phi) = \frac{2}{3}\sqrt{\frac{15}{4\pi}}$ , where $(n_i) = (\sin\theta\cos\phi,\ \sin\theta\ \sin\phi,\ \cos\theta)$. Rotating $Y_2^{+1}(\theta,\phi)$ by an infinitesimal angle ϵ with respect to axis $\vec{e}_z$, we get $Y_2^{+1}(\theta,\phi-\epsilon)$. (Or, in the differential operator language, we are acting $(1-\ \epsilon\vec{e}_z\cdot\vec{r}\times\nabla) = \left(1+\frac{\epsilon\vec{e}_z}{i\hbar}\cdot(-i\hbar\vec{r}\times\nabla)\right)$ to $Y_2^{+1}(\theta,\phi)$) This corresponds to substituting [n] into [$\mathrm{e}_z,-\epsilon$; n] = [e_z,ϵ; n]. The fact that [tensor] is a tensor allows us to "push the arrowheads" as the following way, where the labels "e_z,ϵ" are omitted to avoid clutter.


(175)

Here we used [e_z,ϵ; e^m] = [e^m] + ϵ [e_z; e^m] + $\mathcal{O}(\epsilon^2)$ = $\left(1+(-)^m\epsilon/i\right)$ [e^m] + $\mathcal{O}(\epsilon^2)$. Thus, $Y_2^{+1}(\theta,\phi)$ transforms alike to $\vec{e}^+$, as it gains the phase factor $(1-i\epsilon)$ when subjected to infinitesimal z-rotation.

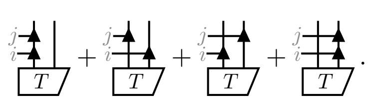
In fact, one can see this by the explicit formula of $Y_2^{+1}(\theta,\phi)$ that is proportional to $e^{i\phi}$ so that $e^{i\phi}\ \to\ e^{i(\phi-\epsilon)} = (1-i\epsilon+\mathcal{O}(\epsilon^2))e^{i\phi}$. However, the strength of tensor language is that we can easily see that this is generalized to arbitrary Y_ℓ^m. By the "arrow pushing," the transformation [n] $\to$ [e_z,ϵ; n] was equivalent to dressing all $\vec{e}^m$'s with outward arrowheads but leaving $\vec{n}$'s as they were. Therefore, $Y_\ell^{+1}(\theta,\phi)$ must have the same transformation property with [e^+], $Y_\ell^{+2}(\theta,\phi)$ must have the same transformation property with [e^+ e^+], and so on.

(d) Consider an $\mathsf{SO}(3)$ transformation of an arbitrary rank-2 tensor [T]. Each index will be dressed with outward arrowheads (rotation matrices): [T] $\to$ [T]. If [arrow] = [line] + $\frac{\epsilon}{i\hbar}$ [n] + $\mathcal{O}(\epsilon^2)$ for a given fixed unit vector $\vec{n}$, that is,

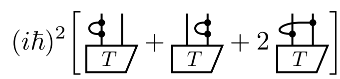

(176)

Setting this to zero gives the infinitesimal version of rotational invariance. (We apologize for using a different symbol [i] for denoting the generators of rotation, $i\hbar$ [i], instead of the earlier [S, i] for the sake of visual brevity.) One can think the bracketed term in the right hand side of Eq. (176) as a result of acting $\hat{S}_i$ to [T], because it measures how [T] deviates from itself when infinitesimally rotated.

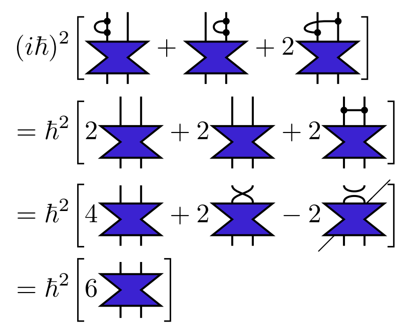
Acting $\hat{S}_i$ then $\hat{S}_j$ to [T], we get


(177)

Contracting this with δ_{ij} and substituting [i] $= i\hbar$ [i], we obtain $\hat{S}^2 = \hat{S}_i\hat{S}_i = \delta_{ij}\hat{S}_i\hat{S}_j$ applied to [T].

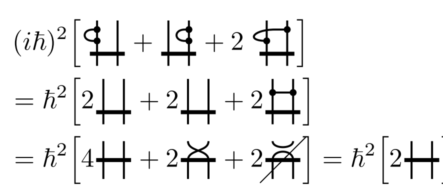

(178)

In general, we cannot determine the value of Eq. (178) for an arbitrary rank-2 tensor T_{ij} = [i j; T]. However, its traceless and symmetric part, $\frac{2}{3}$ [tensor; T], is an eigenstate of $\hat{S}^2$ having an eigenvalue $2(2+1)\hbar^2 = 6\hbar^2$. To see this, it suffices to show that the answer of this problem is $6\hbar^2$ [tensor].


(179)
(180)
(181)
(182)

This justifies to call the symmetric traceless part of a rank-2 tensor by "spin-2 part!"

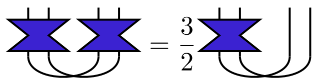
How about the trace-only part? Surely, there are no rooms for attaching [i] to a scalar, so it will have an eigenvalue 0 for $\hat{S}^2$. For the antisymmetric part,

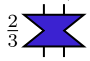
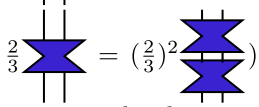
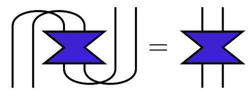

(183)
(184)
(185)

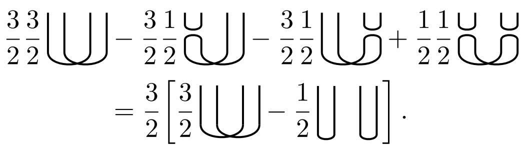
We obtain the eigenvalue $1(1+1)\hbar^2$; thus, it is indeed a "spin-1 part." Note that rank-1 tensors (vectors) also have an eigenvalue $1(1+1)\hbar^2$ for $\hat{S}^2$; the secret underlying here is the arrow pushing [diagram] $= -\frac{1}{2}$ [diagram]. Considering the infinitesimal version of this arrow pushing, one can confirm that applying $\hat{S}^2$ in the upper two indices of [diagram] $= -\frac{1}{2}$ [diagram] is identical to applying $\hat{S}^2$ in the middle internal line, which will give $(i\hbar)^2$ [diagram] $= 2\hbar^2$ [line].

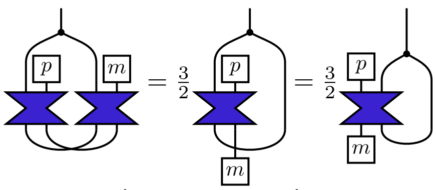
Lastly, if you are interested, try applying $\hat{S}^2$ to the spin-ℓ representation [tensor] with δ_{ii} = ◯ $= d$. This will give the Casimir of spin-ℓ representation of $\mathsf{SO}(d)$, $\ell(\ell+d-2)$.

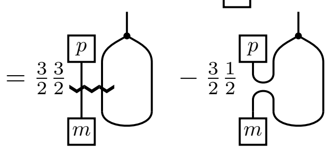
A67 Before we proceed, it is fruitful to consider the following subdiagram first.

 (186)

Eq. (186) is easily confirmed if one understands that  is the "symmetric traceless projector" (so that ) and . Or, one can see it by direct calculation:

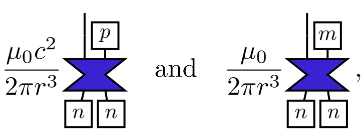
 (187)

(a) 

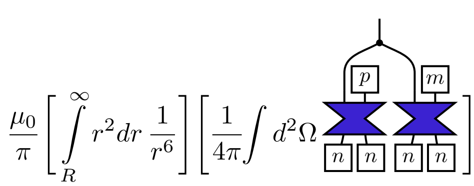


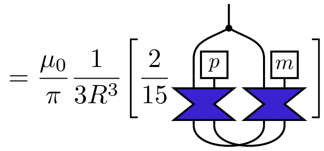
$= \frac{1}{2}(0 + \frac{3}{2}\frac{3}{2}\vec{m}\times\vec{p}) - \frac{3}{2}\frac{1}{2}\vec{p}\times\vec{m} = \boxed{\frac{15}{8}\vec{m}\times\vec{p}}$.

Consider a ball of radius R, uniformly polarized with total electric and magnetic dipole moments $\vec{p}$ and $\vec{m}$. What would be the electromagnetic momentum of this system?[13] The angular integration of this problem can be worked out algebraically by tensor methods. Contribution from the inside of the ball is $\frac{1}{\mu_0 c^2}\left((-\frac{\mu_0 c^2}{4\pi R^3}\vec{p}) \times (\frac{2\mu_0}{4\pi R^3}\vec{m})\right)\frac{4\pi}{3}R^3 = \frac{\mu_0}{\pi R^3}\frac{1}{6}\vec{m}\times\vec{p}$. For the outside of the ball, where the electric and magnetic fields are given by

 (188)

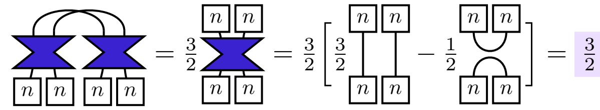
respectively (Eqs. (27) and (28)), we have

 (189)

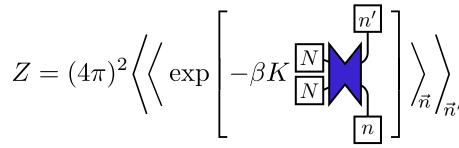
 (190)

$$= \frac{\mu_0}{\pi}\frac{1}{3R^3}\left[\frac{2}{15}\frac{15}{8}\vec{m}\times\vec{p}\right] = \frac{\mu_0}{\pi R^3}\frac{1}{12}\vec{m}\times\vec{p}. \quad (191)$$

Eq. (168) is used when obtaining Eq. (190). Finally, the answer is $\frac{\mu_0}{\pi R^3}(\frac{1}{6}+\frac{1}{12})\vec{m}\times\vec{p} = \frac{\mu_0}{4\pi R^3}\vec{m}\times\vec{p}$.

(b) 

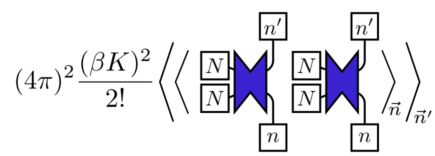
Consider two point dipoles $\vec{\mu}_1$ and $\vec{\mu}_2$ in thermal equilibrium at an inverse temperature β. The dipole moments can orient all directions and the separation between the two is fixed at $\vec{R} := R\vec{N}$. The canonical partition function is equal to the following, where K is a constant that measures the coupling between two dipoles ($2\frac{\mu_0\mu_1\mu_2}{4\pi}$ if the dipoles are magnetic), $\vec{\mu}_1 := \mu_1\vec{n}$, and $\vec{\mu}_2 := \mu_2\vec{n}'$.

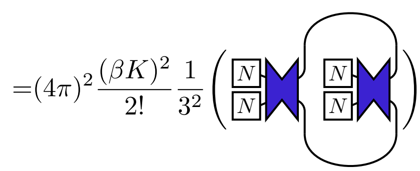
 (192)

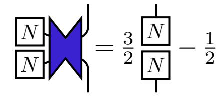
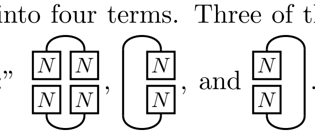
$\langle\cdots\rangle_{\vec{n}}$ and $\langle\cdots\rangle_{\vec{n}'}$ denote spherical averaging with respect to solid angle measure corresponding to $\vec{n}$ and $\vec{n}'$, respectively. From this partition function, we can obtain the average force between the two dipoles, which will result in Lenard-Jonnes potential.

What would be the leading nontrivial term? A typical calculation that employs coordinates θ, ϕ, θ', and ϕ' goes pages long,[35] yet obtaining the next term in such way by hand will be much more complicated. However, with the aforementioned algebraic (or combinatoric) spherical integrating and the graphical notation, even a general formula for terms of arbitrary orders is easily obtainable by a "hairband cutting game."

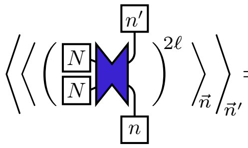
First, consider the β^2 term in the expansion of the exponential Eq. (192). (Notice that terms of odd powers of β vanishes.)

 (193)

 (194)

We have already calculated the bracketed term in Eq. (194) in the main problem: it was $\frac{3}{2}$. However, let us revisit it in terms of the "hairband cutting game." Recall that . Substituting this, the term within the parentheses in Eq. (194) expands into four terms. Three of them "breaks the hairband:" . The remaining one does not. If a "hairband" is broken, its value is 1 ($\vec{N}\cdot\vec{N}$ or $(\vec{N}\cdot\vec{N})^2$). If it is not, its value is 3 (δ_{ii}). Thus, the bracketed term in Eq. (194) is equal to $1 \cdot (\frac{3}{2}\frac{3}{2} + \frac{-1}{2}\frac{3}{2} + \frac{3}{2}\frac{-1}{2}) + 3 \cdot (\frac{-1}{2}\frac{-1}{2}) = \frac{3+3}{4} = \frac{3}{2}$.

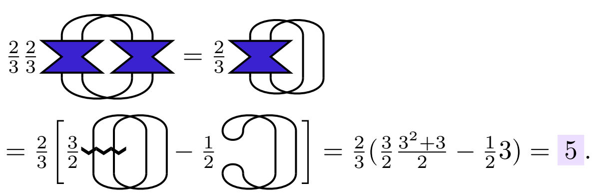
The graphical notation effectively helps doing this "hairband combinatorics"—counting the possibilities of getting broken or intact hairbands—for arbitrary orders of β. The resulting formula is the following.

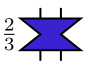


$$= \frac{1 + 2^{1-2\ell} p_\ell(3)}{(2\ell+1)^2}, \qquad (195)$$

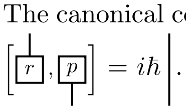
$$p_\ell(d) := \begin{cases} 1 & (\ell = 1) \\ 1 + \sum_{j=1}^{\ell-1} \binom{\ell}{j} d^{2j-1}(d-2j) & (\ell > 1) \end{cases}$$

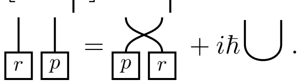
$p_2(3) = 7$, $p_3(3) = -71$, and so on; Eq. (195) is equal to $\frac{1}{2^2}\frac{3}{2}$, $\frac{1}{2^4}\frac{30}{25} = \frac{1}{2^4}\frac{6}{5}$, and $\frac{1}{2^6}\frac{-78}{49}$ for $\ell = 1$, 2, and 3, respectively. Meanwhile, obtaining the general formula of higher-order terms by integration introducing angular coordinates is virtually an impossible task by hands. This example serves as a instructive demonstration for diagrammatic perturbation.

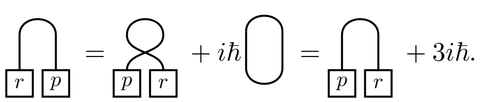
(c) 

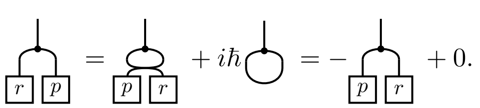
This is the dimensionality of the spin-2 projector,  (recall the combinatorics in **P50**).

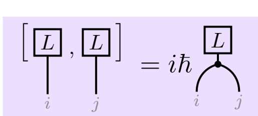
D. Addendum

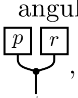
A68 The canonical commutation relation, $[x_i, p_j] = i\hbar\delta_{ij}$, reads . Let us employ this in a form of Eq. (33),

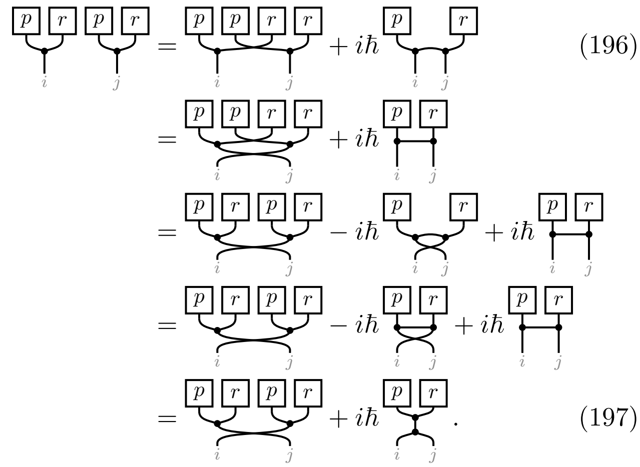


(a) 

(b) 

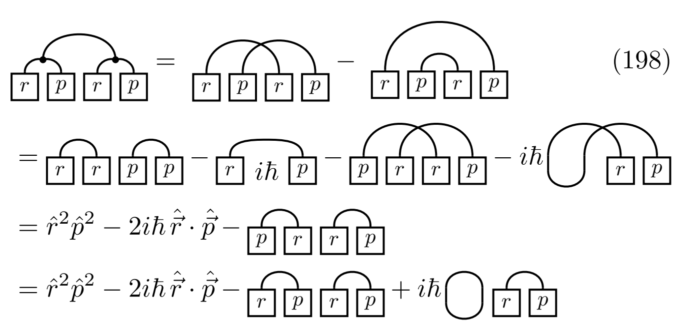
A69  .

The orbital angular momentum, $\hat{L}_i = \epsilon_{ijk}\hat{x}_j\hat{p}_k = -\epsilon_{ijk}\hat{p}_j\hat{x}_k =$ , is an implementation of this:



This completes the proof. The last step is due to the Jacobi identity (**P16**).

A70 Connecting the two ends of Eq. (196) by a Kronecker delta,

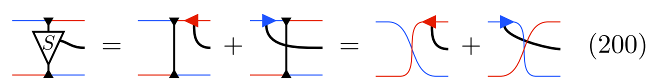


$$= \hat{r}^2\hat{p}^2 - 2i\hbar\,\hat{\vec{r}}\cdot\hat{\vec{p}} - (\hat{\vec{r}}\cdot\hat{\vec{p}})^2 + 3i\hbar\,\hat{\vec{r}}\cdot\hat{\vec{p}}$$

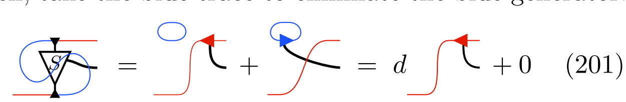
$$= \hat{r}^2\hat{p}^2 - (\hat{\vec{r}}\cdot\hat{\vec{p}})^2 + i\hbar\,\hat{\vec{r}}\cdot\hat{\vec{p}} \qquad (199)$$

A71 This instructive problem is based on the $\mathsf{so}(1,3) \cong \mathsf{su}(2) \oplus \mathsf{su}(2)$ algebra. For more precise treatment that distinguishes upper and lower indices, see, e.g., Srednicki.[36]

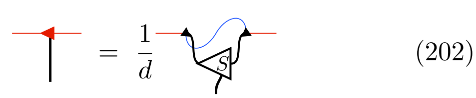
Use completeness first.

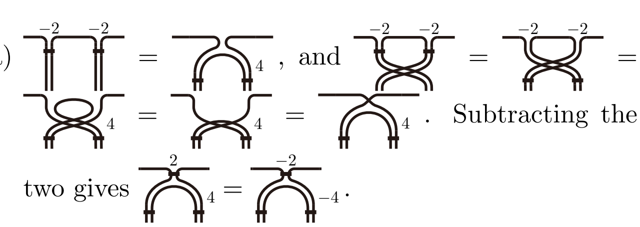


Then, take the blue trace to eliminate the blue generator.



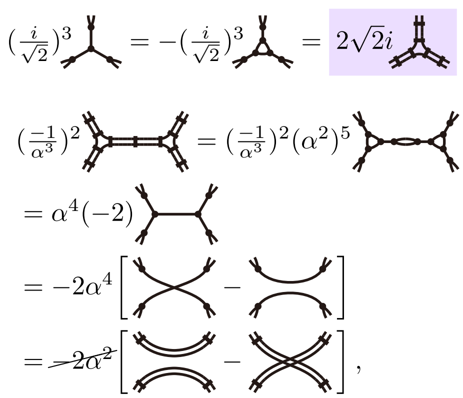
Massaging the both sides, we get the answer.



A72 (a) 

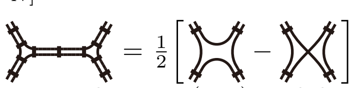
(b) The lesson from **P49**b is that an I-line can be converted to an i-line by the cross product machine and vice versa. Refer to Eq. (132). The converters were $\frac{1}{2!}$ and $-$; this is a standard convention in tensor calculus distinguishing contravariant and covariant indices (up-down hierarchy). Meanwhile, a different normalization, $\frac{i}{\sqrt{1!2!}}$ and $\frac{i}{\sqrt{1!2!}}$ is more apt for the "heterarchized" calculus.

Then, ϵ_{ijk} with each of its indices converted to I-indices perfectly copies the syntax of ϵ_{ijk} to the I-world. As demonstrated in Eq. (205) to Eq. (207), an I-world expression is translated into an i-world epsilon network by the converters and brought back to the I-world after simplification.



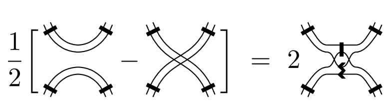
(203) (204) (205) (206) (207)

where $\alpha := \frac{i}{\sqrt{2}}$. The rightmost expression in Eq. (203) is the answer for this problem with its complex conjugate, because we could have used $\frac{-i}{\sqrt{1!2!}}$ and $\frac{-i}{\sqrt{1!2!}}$ as converters. The resulting ϵ_{IJK} satisfies total antisymmetry $\epsilon_{IJK} = -\epsilon_{JIK} = \cdots$ and $\epsilon_{IJK}\epsilon_{KLM} = \delta_{IL}\delta_{JM} - \delta_{IM}\delta_{JL}$. (Also $\epsilon_{IJK}\epsilon_{LMN} = 3!\delta^I_{[L}\delta^J_M\delta^K_{N]}$; check it!)

Note that  not in general is proportional to Eq. (207). While transforming Eq. (205) into Eq. (207), we used the fact that is equal to $\frac{-1}{2}$ so that it can be "factorized" into two tripod totally antisymmetric tensors, which is not true in dimensions other than three.

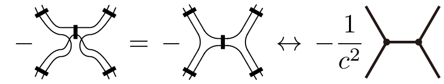
One more comment: we could have determined ϵ_{IJK} by first setting it to be c then finding the value of c. Then, following the same calculation in Eq. (205) to Eq. (207), the condition for to be reproduced turns out to be $c^2 = -8$. We employ this approach in the next problem.

(c) Let $\leftrightarrow c$. c is a constant to be determined. First, let us impose the condition that it reproduces . Note that



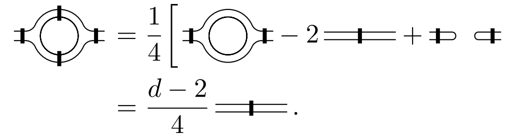
(208)

holds generally (cf. the internal wiring in Eq. (43)). Then, from the binor identity $+ 2 = 0$, Eq. (208) is equal to



(209)

Therefore, $c^2 = 2$; $c = \sqrt{2}$ or $c = -\sqrt{2}$. Next, let us demand that $c^2 = -2$. Using only $= d$ and the definition of the antisymmetrizer, we have



(210) (211)

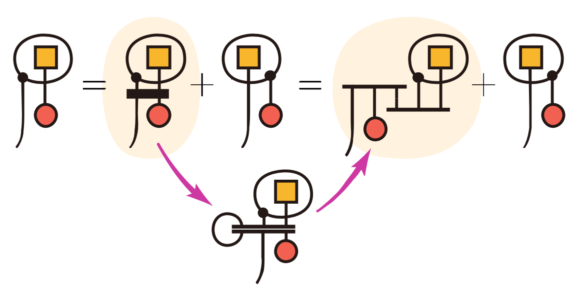
Therefore, the equation $c^2(d-2) = -8$ must be satisfied. In the previous problem, $c^2 = -8$ so that $d = 3$ ($\delta_{ii} = 3$). In case of binors, $c^2 = 2$ so that $d = -2$!

As $d = \delta_{aa}$, it seems that binors are tensors living in a (-2)-dimensional space: $\mathsf{SO}(-2)$-tensors. In this sense, Penrose[17] introduced the idea of "negative dimensional tensors." The reader might find it uncomfortable to accept. Practically, this negative dimensionality can be understood as a consequence of "heterarchizing" spinors. The binor language can be repackaged into spinor algebra, where the converter between binors and $\mathsf{SO}(3)$-vectors being the Pauli matrices. Also, the distinction between contravariant and covariant indices is restored, such as $\rightarrow -$ $= -$ $= -\epsilon_{ab}\epsilon^{ab} = \epsilon_{ab}\epsilon^{ba}$ and $= -\frac{1}{2}$ $\rightarrow$ $= +\frac{1}{2}$ ($\epsilon^{ab}\epsilon_{cd} = \delta^a_c\delta^b_d - \delta^a_d\delta^b_c$). Since $\epsilon_{ab}\epsilon^{ba} = -2$, we have a negative d. Meanwhile, while restoring hierarchy, one observes that symmetrization and antisymmetrization changed their role. Cvitanović and Kennedy[16] discussed on this point and clarified the meaning of tensors of negative dimensions.

Antisymmetrizing two d-dimensional indices, we obtain a collective index of dimensionality $\binom{d}{2}$. $d = 3$ produces the self-similar infinite tower: 3, $\binom{3}{2} = 3$, $\binom{\binom{3}{2}}{2} = 3$, $\cdots$. (It is the "critical point.") If we start from binors, we have -2, $\binom{-2}{2} = 3$, $\binom{\binom{-2}{2}}{2} = 3$, $\cdots$. Once we entered the three dimensions, we will always stay there. Thus, problems (b) and (c) complete the exploration of the stairs of the recursive tower.

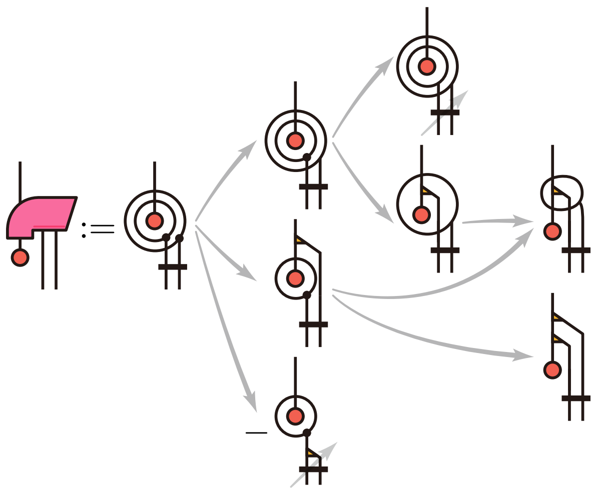
Bonus Cuts

Listen to Tensors



https://soundcloud.com/joonhwi-kim/listen-to-tensors

Riemann Tensor



References

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1 R. Penrose, in *Quantum Theory and Beyond: Essays and Discussions Arising from a Colloquium*, edited by T. Bastin (Cambridge University Press, 1971) Chap. 14, pp. 151–180.

2 G. E. Stedman, *Diagram techniques in group theory* (Cambridge University Press, Cambridge, 2009).

3 P. Cvitanović, *Group theory: birdtracks, Lie's, and exceptional groups* (Princeton University Press, 2008) p. 273.

4 We do not distinguish the SO(3) contravariant and covariant indices in this article; the use of upper indices here is just for notational convenience.

5 In fact, we also need the fact that the only rotationally invariant rank-2 tensor is the Kronecker delta.

6 See footnote 18 of the main article.

7 B. Coecke, in *AIP Conference Proceedings* (AIP, 2006) pp. 81–98.

8 H. Taouk, *Wave propagation in general anisotropic media*, Master's thesis, Ohio University (1986).

9 R. Penrose, *The road to reality: A complete guide to the physical universe*, vintage ed. (Vintage Books, New York, 2007).

10 If you are sad that the "hierarchical" design makes the previously dancing diagrams "stiffer," try another way: assigning a "hook" or an "eye"[11] to each end to the lines. For example, $B_i = \boxed{B}$——▫ $_i$, $A^i =$ $_i$▪——$\boxed{A}$, $B_i A^i = \boxed{B}$——▪——$\boxed{A}$, $\varepsilon_i{}^{jk} =$ $_i$ ⅄ ^k_j, etc. Hooks represent output lines; eyes represent input lines. (This is equivalent to making the lines directed.)

11 I. V. Lindell, *Differential forms in electromagnetics* (John Wiley & Sons, 2004).

12 The word "golden" is put just for linguistic playfulness, as a reminiscence of D. R. Hofstadter's book *Gödel, Escher, Bach: an eternal golden braid*.

13 D. J. Griffiths, *Introduction to electrodynamics* (Prentice Hall, 1999), p. 576.

14 D. J. Griffiths, American Journal of Physics, **50**, p. 698 (1982).

15 Two lines of the same type related by a basis change can be connected by the matrix that corresponds to the basis change.

16 P. Cvitanovic and A. D. Kennedy, Physica Scripta **26**, p. 5 (1982).

17 R. Penrose, Combinatorial mathematics and its applications, pp. 221 (1971).

18 P. Ball, *Flow: Nature's patterns: a tapestry in three parts* (OUP Oxford, 2009).

19 Non-local fields can be also considered, in principle.

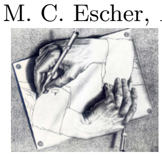
20 For a clear construction of Feynman diagrams in the functional calculus language, refer to N. Beisert's lecture notes.[21]

21 N. Beisert, Quantum Field Theory II, Lecture Notes, (2014). Retrieved from `http://edu.itp.phys.ethz.ch/fs13/qft2/`.

22 D. Simmons-Duffin, arXiv e-prints, (2016), arXiv:0910.1362 [hep-th].

23 For another example, think about doing braided group calculations by anyons ("topological quantum computing").

24 M. C. Escher, *Drawing Hands*, 1948. Lithograph.



25 This is the "arrowheads as abstract indices" notation that the first author suggests.[26] It is a compromise plan between graphical and plaintext notations. One-forms and vectors are denoted as $\overleftarrow{W}$ and $\overrightarrow{V}$, respectively, and their contraction is written as $\overleftarrow{W}\overrightarrow{V}$ (cf. the contraction of a bra and a ket $\langle\beta|\alpha\rangle$). A $\binom{1}{1}$-tensor will be denoted as $\overleftrightarrow{T}$, which has "one output and input terminals." It acts on a vector $\overrightarrow{V}$ to give a vector $\overleftrightarrow{T}\overrightarrow{V} = \overrightarrow{TV} = \overrightarrow{e}_i T^i{}_j V^j$. This notation is used exclusively in this problem.

26 J.-H. Kim and J. Nam, arXiv e-prints, (2019), arXiv:1912.11485 [physics.ed-ph].

27 E. Peterson, arXiv e-prints, (2009), arXiv:0910.1362 [math.HO].

28 E. Peterson, arXiv e-prints, (2007), arXiv:0712.2058 [math.HO].

29 J. D. Romano and R. H. Price, American Journal of Physics, **80**, pp. 519 (2012).

30 Moreover, further invariant products between a symmetric traceless rank-2 tensor and another invariant object (a scalar, a vector,[31] or symmetric traceless rank-2 tensor, etc.) can be defined. Doing this requires irreducible decomposition formulae of higher-rank tensors, such as the decomposition in **A20** and the techniques in Section I C 6.

31 M. Lazar, Journal of Applied Mathematics and Mechanics, **96**, 11, pp. 1291–1305 (2016).

32 In fact, one can take this decomposition as a starting point then construct the invariant symbols, figuring out what projector of an invariant subspace satisfies the "proper propagation of arrows" condition and identifying it with a projector made by generators of SO(3), as demonstrated in Cvitanovič's Group Theory.[3] The validity of Eq. (26) can be confirmed without prior knowledge of invariant symbols by the orthogonality of the spin-0, 1, and 2 projectors in Eq. (26) and the fact that their dimensionalities sum to $1 + 3 + 5 = 3 \times 3$ (cf. **A67**c).

33 J. D. Jackson, *Classical Electrodynamics* (Wiley, 1999), p. 808.

34 D. J. Griffiths and D. F. Schroeter, *Introduction to quantum mechanics* (Cambridge University Press, 2004).

35 S. B. Cahn, G. D. Mahan, and B. E. Nadgorny, *A Guide to Physics Problems: Part 2: Thermodynamics, Statistical Physics, and Quantum Mechanics* (Springer Science & Business Media, 1997), pp. 131–133.

36 M. A. Srednicki, *Quantum Field Theory* (Cambridge University Press, 2007), p. 641.

37 H. Elvang and Y.-T. Huang, *Scattering Amplitudes in Gauge Theory and Gravity* (Cambridge University Press, 2015).

38 Z. Bern and J. J. M. Carrasco, J. Henrik, Physical Review D, **78**, 8: 085011 (2008).

39 N. Arkani-Hamed and Y. Bai, S. He and G. Yan, Journal of High Energy Physics, **8**, p. 40 (2018).

40 N. E. J. Bjerrum-Bohr, P. H. Damgaard, R. Monteiro, and D. O'Connell. *Journal of High Energy Physics*, **6**, 61 (2012).

41 C.-H. Fu, Y.-J. Du, and B. Feng. *Journal of High Energy Physics*, **3**, 50 (2013).

42 C.-H. Fu, P. Vanhove, and Y. Wang. *Journal of High Energy Physics*, **9**, 141, 2018.

43 R. Monteiro and D. O'Connell. *Journal of High Energy Physics*, **7**, 7 (2011).