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Graphical Notation for Vector Calculus and Its Generalization

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벡터 미적분학의 그래프적 표기법과 그 일반화

Graphical Notation for Vector Calculus and Its Generalization

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by

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Abstract

Graphical notation has been introduced and utilized in physics for years, but it still remains as relatively minor notation compared to the conventional plaintext notation despite its effectiveness, readability, and applicability. In this thesis, the graphical notation for vector calculus is reviewed and generalized.

The essence and advantage of the graphical notation are illustrated with vector calculus as a self-contained example. Basic rules are introduced and utilized to prove some representative vector calculus identities, illustrating how concise and effective the graphical notation is. More generalized concepts and notations, such as the algebra of gamma matrices and quantum mechanical commutation relation, are reviewed to show how the notation can be modified and extended.

By enabling graphical notation to be readily applied and taught in undergraduate level vector calculus, the entry barrier of graphical notation would be lowered and the practical use of the notation would be promoted. The generalization examples would encourage readers to manipulate and invent graphical notations by themselves when needed, enlarging and diversifying the use of graphical notation in general. This would lead to more concrete and efficient understanding and presentation of physics.

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Chapter 1. Introduction

Notation is closely related to how to conceive concepts and utilize them. Vector notation is an representative example worth mentioning. The modern vector notation was invented by Heaviside, modifying the concept of quaternions.[10] Before that, electromagnetic laws were written for each Cartesian component so that there were a lot more variables and equations than what is now being commonly taught (see p. 486 of [16]), which makes it hard to grasp the meaning of them. In that sense, presenting his concise notation, Heaviside himself wrote that "in the Cartesian method, we are led away from the physical relations" whereas in vectorial algebra, vectors and their mutual relations "are usually exhibited in a neat, compact, and expressive form, whose inner meaning is evident at a glance to the practiced eye." (p. 133 of [10])

There is a similar story regarding tensorial notation, but in a different form. Penrose's graphical notation is a way to express tensorial structures as a graph-like diagrams composed of lines, nodes and more. In this notation, the expression is often more concise and convenient to handle, compared to conventional plaintext notation. It seems that he devised the notation while dealing with spin network and binor algebra, showing the isomorphism between expressions in binor diagrams and those written in two-dimensional tensor expressions.[22] He greatly extended and utilized the notation throughout his publications, especially to tensorial algebra and calculus.[18, 19, 20, 17, 21] Meanwhile, it needs to be mentioned that there had been similar graphical approaches before him as well.[29, 30, 15]

Since late 20th century, the use of graphical notations has been accelerated not only for tensor algebra and calculus but also other disciplines, such as group theory[7, 12, 9, 28], linear algebra[24, 25, 26, 27], quantum mechanics[6, 3, 2, 5, 1, 4], vector algebra[28] and vector calculus[13, 14]. The key idea remains the same; indices in plaintext notation are replaced with lines, getting rid of the need to specify every single index and consequently making equations more graspable and manipulatable as a whole picture. In other words, graphical notation liberates us from one-dimensional written instructions to two-dimensional drawn illustration.

Meanwhile, interestingly enough, vector calculus with graphical notation had not been covered in literatures until very recently[13, 14] whereas all the other works has proceeded. The reason of this late application, despite its applicability, might be attributed to the lack of interest and demand, difficulty of drawing diagrams for publication, etc. Some might even conceive that vector calculus does not need any new tools and it is an overkill to introduce graphical notation for vector calculus.

Nonetheless, the author claims that it is beneficial to introduce and spread the notation in early stage of education so that students can adapt themselves to diverse tools and languages and get benefits from each of them. They would find it much familiar and graspable when they see other similar graphical approaches in more advanced topics.

To that end, in this thesis, the graphical notation on vector algebra and calculus are reviewed and some applications and generalizations of the notation are covered, including the algebra of gamma matrices and quantum mechanical commutation relation. This way of introduction would help readers get familiarized to graphical notation and utilize it skillfully.

Chapter 2. Review of Vector Calculus with Graphical Notation

This section introduces much of concepts and figures of the author's previous publications.[13, 14] For more detailed discussion, please consult with them.

2.1 Basic Rules

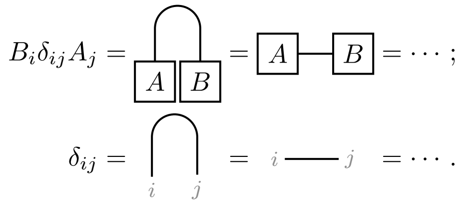
As briefly mentioned in the last section, the key of graphical notation is to replace indices with lines. The names of quantities (such as E for electric field and B for magnetic field) are put in boxes. As we will see, this greatly reduces the visual complexity and bulkiness of equations, especially when there are many entangled terms. To begin with, basic notations for scalars, vectors, and their simple arithmetic operations are expressed as follows:

$$\begin{aligned}
\text{Scalars:}\quad & f = \boxed{f} \\
\text{Vectors:}\quad & \vec{A} = \text{---}\boxed{A} \\
\text{Scalar multiplication:}\quad & fg = \boxed{f}\ \boxed{g} \\
& f\vec{A} = \text{---}\boxed{A}\ \boxed{f} \\
\text{Addition/subtraction:}\quad & f \pm g = \boxed{f} \pm \boxed{g} \\
& \vec{A} \pm \vec{B} = \text{---}\boxed{A} \pm \text{---}\boxed{B}
\end{aligned} \tag{2.1}$$

Note that a vector has one attached line because it has one index, whereas a scalar has no attached line. For their basic arithmetic notations, we simply follow conventional notation. When two objects are juxtaposed to imply multiplication, their relative order or position has no significance. Namely, $\boxed{f}\,\boxed{g} = \boxed{f}^{\boxed{g}} = \begin{matrix}\boxed{g}\\ \boxed{f}\end{matrix} = \cdots$ etc.

2.2 The Inner Product and the Kronecker Delta

There are two representative vector products; the inner product and the cross product. The inner product between two vectors $\vec{A}$ and $\vec{B}$ is defined as $\vec{A} \cdot \vec{B} := A_i B_j \delta_{ij} = A_i B_i$, with Einstein summation convention. This implies the same tail (index i) is shared by two vector quantities, so we can easily justify the inner product is denoted by $\boxed{A}\text{—}\boxed{B}$. On the other hand, thinking about the last equality, $A_i B_j \delta_{ij} = A_i B_i$, gives a justification for the notation of the Kronecker delta. The nature of the Kronecker delta is "changing the name of index into the other one", or "connecting two different indices". In that sense, we can find a proper notation of the Kronecker delta, which is a connecting line.



(2.2)

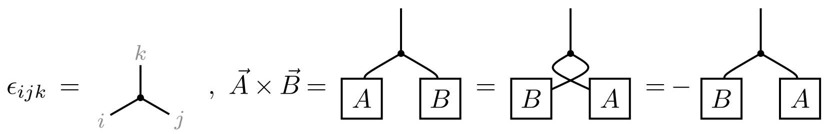
(2.3)

Note that the index marker, which is written in gray color, can be either turned on for clarification of index or off for convenience. When it is turned off, the relative positions of line terminals are presumed to carry the index information.

2.3 The Cross Product, the Levi-Civita Symbol, and the Scalar Triple Product

We found the notation of the inner product and the Kronecker delta naturally, following the key idea of graphical notation. For the cross product and the Levi-Civita symbol, however, we need to devise notations for them. At first glance, one might think a box named epsilon with three attached lines would work and it is fine to do so. Past literatures, however, have introduced a designaated notations for them because of three reasons. First, the cross product (and the Levi-Civita symbol as well) has special properties regarding index permutation. Second, the Levi-Citiva symbol is a constant quantity–not a function of space coordinates. Lastly, it is used very frequently so that it is convenient to denote them in a simpler way, rather than drawing boxes every time. The author sticks to the tripod-shaped convention because of its simplicity, which has been introduced and used widely. A more general notation for n-dimensional Levi-Civita symbol will be introduced in Ch. 4.2.

The cross product between two vectors is defined as $(\vec{A} \times \vec{B})_k = A_i B_j \varepsilon_{ijk}$. First, it is an operation that takes two input vectors and gives one output vector as a whole. Second, the operation itself is antisymmetric; $\vec{B} \times \vec{A} = -\vec{A} \times \vec{B}$, or equivalently $\varepsilon_{jik} = -\varepsilon_{ijk}$. Lastly, it is symmetric under cyclic permutation of indices (or equivalently under even permutation of indices), $\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij}$. Considering these, the following notation has been used widely for simplicity and applicability. The notation intrinsically involves the condition that *the connected lines should be read counterclockwise from the center.*



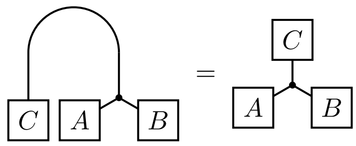
(2.4)

If the Levi-Civita symbol( ) is to be read clockwise, one should keep a minus sign in their mind because of its antisymmetricity. Or, equivalently, if any two lines are swapped in the graph, the term should be negated. Intuitively speaking, the lines are rigid near the center, so it generates negating impact when two lines are swapped. Note that mere crossing of lines, without changing the connectivity between lines and vectors (or indices), has no negating impact because we still read the order of connected lines the same. Alternatively, crossing of lines can be thought as mere extension of lines with the Kronecker deltas ($\varepsilon_{ijk} = \varepsilon_{ilm} \delta_{lj} \delta_{mk}$).

Note that the codified rules for the products and symbols are implemented as the way how their graphical representations are to be read. For the Kronecker delta, one end cannot be distinguished from the other end which means $\delta_{ij} = \delta_{ji}$, or equivalently $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$. For the Levi-Civita, it should be read counterclockwise, so that $\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij}$, and negated if read clockwise, so that $\varepsilon_{jik} = -\varepsilon_{ijk}$

The scalar triple product is a combination of the dot product and the cross product. The scalar triple product of three vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ is defined to be $\left[\vec{A}, \vec{B}, \vec{C}\right] = \vec{A} \cdot (\vec{B} \times \vec{C})$ and it has a cyclic symmetry of $\left[\vec{A}, \vec{B}, \vec{C}\right] = \left[\vec{B}, \vec{C}, \vec{A}\right] = \left[\vec{C}, \vec{A}, \vec{B}\right]$. This can be noticed by merely looking at the graphical representation of it. Depending on where you start to read the product, or if you rotate the whole graph, it is more than evident that the scalar triple product has the cyclic symmetry, which is embodied as the rotational symmetry of the graph.

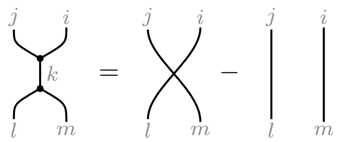
$$\left[\vec{A}, \vec{B}, \vec{C}\right] = \vec{A} \cdot \left(\vec{B} \times \vec{C}\right) = \tag{2.5}$$



$$= \left[\vec{B}, \vec{C}, \vec{A}\right] = \left[\vec{C}, \vec{A}, \vec{B}\right] \tag{2.6}$$

2.4 The Contracted Epsilon Identity and The Vector Triple Product

The contracted epsilon identity is one of the core tools of vector algebra and calculus. It can be formally proven with graphical notation, but let us take it for granted here (refer to P17 and P18 of [14], if interested in). The graphical representation of the contracted epsilon identity is as follows:

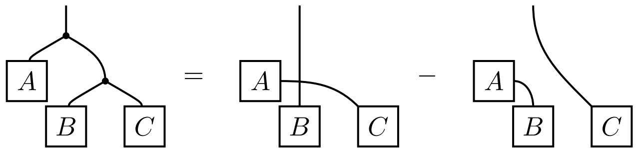


(2.7)

$$\updownarrow$$

$$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{jm}\delta_{il} - \delta_{jl}\delta_{im}$$

Most of vector algebraic identities can be proven using this. Again, compared to the plaintext notation, what the graphical notation provides is a fast and obvious approach. Without translating everything into plaintext notation and looking into every single index, it clearly shows internal algebraic structure and enables us to proceed algebra efficiently. As an example, let us graphically represent the vector triple product of three vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ and apply the identity.



$$\updownarrow$$

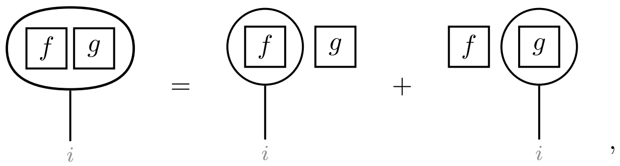
$$\vec{A} \times \left(\vec{B} \times \vec{C}\right) \quad = \quad \vec{B}(\vec{A} \cdot \vec{C}) \quad - \quad \vec{C}(\vec{A} \cdot \vec{B}) \tag{2.8}$$

For more such graphical proofs of vector algebra identities, please refer to I. A. of [14].

2.5 The Differentiation

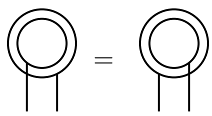
What previous publications on graphical representation of vector identities (e.g. [28]) have not covered is vector calculus identities, which involves differentiation of fields. Following Penrose's convention, a balloon-shaped loop with one attached line is a good way to represent differentiation, ∂_i, because it is easy to denote and recognize what it is differentiating. The role of the attached line is to specify the differentiation index.

There are two rules for differentiation; one is the product rule, $\partial_i(fg) = \partial_i(f)g + f\partial_i(g)$, and the other is the commutativity of partial differentiation, $\partial_i\partial_j = \partial_j\partial_i$. Each of these rules are graphically represented as follows:



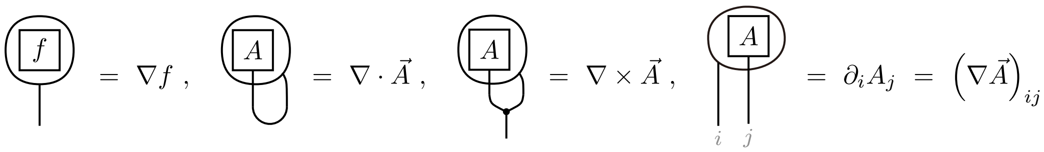
$$\updownarrow$$

$$\partial_i(fg) \quad = \quad \partial_i(f)\,g \quad + \quad f\,\partial_i(g) \tag{2.9}$$



$$\leftrightarrow \quad \partial_i\partial_j \;=\;= \partial_j\partial_i. \tag{2.10}$$

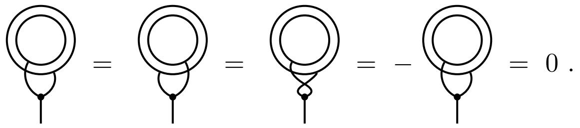
Some examples of differentiation-involved expressions are as follows:



$$= \nabla f\ , \quad = \nabla\cdot\vec{A}\ , \quad = \nabla\times\vec{A}\ , \quad = \partial_i A_j = \left(\nabla\vec{A}\right)_{ij} \tag{2.11}$$

Note that the gradient of a vector, $\nabla\vec{A}$, is naturally expressed. This is the illustration of one of the benefits the graphical notation brings. The concept of tensors, a potentially tricky concept to students, can be clearly and smoothly introduced.

Meanwhile, it is worth mentioning that any second order differentiation contracted with the Levi-Civita symbol is itself zero, because of the commutativity of partial differentiation (symmetry under changing two indices) and the antisymmetricity of the Levi-Civita symbol. This is relevant for following identities: $\nabla \times (\nabla f) = 0$, $\nabla \cdot (\nabla \times \vec{A}) = 0$.

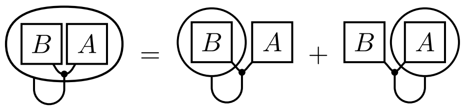


(2.12)

The result is zero because it is the same as itself negated. The first equality comes from the commutativity of derivatives, Eq. 2.10, and the third equality comes from the antisymmetry of the cross product, Eq. 2.4.

2.6 Proofs of Some Vector Calculus Identities

Vector calculus identities can be proven by the combination of the rules that have been introduced so far. One of the simplest cases is $\nabla \cdot (A \times B)$. Applying the product rule and reading the result properly, the identity can easily be proven as follows. Note that the first term on the right-hand side is read reversely (clockwise) and negative sign came out:

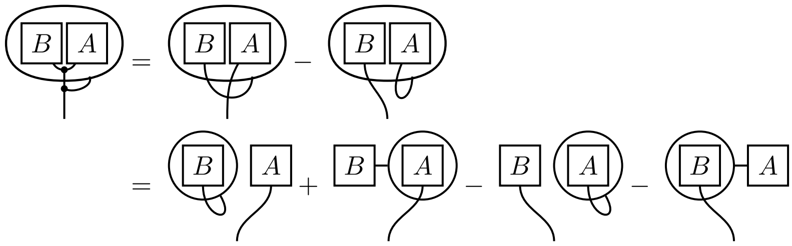


(2.13)

$$\updownarrow$$

$$\nabla \cdot (\vec{A} \times \vec{B}) = -(\nabla \times \vec{B}) \cdot \vec{A} + (\nabla \times \vec{A}) \cdot \vec{B}$$

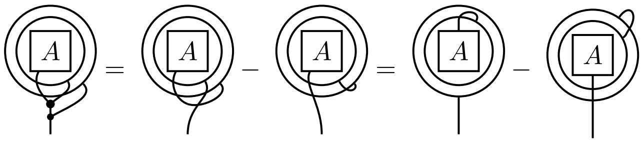
Other two examples worth mentioning are $\nabla \times (A \times B)$ and $\nabla \times (\nabla \times A)$. The former needs the contracted epsilon identity and the product rule for its proof. The latter needs the contracted epsilon identity and the commutativity of partial differentiation. These examples illustrate the usefulness of graphical notation particularly well, because otherwise it takes some time to translate the vectorial expression into plaintext notation, proceed the calculation, and translate it back to vectorial expression.



(2.14)

$\updownarrow$

$$\nabla \times (\vec{A} \times \vec{B}) = (\nabla \cdot \vec{B})\vec{A} + \vec{B} \cdot \nabla\vec{A} - \vec{B}(\nabla \cdot \vec{A}) - \vec{A} \cdot \nabla\vec{B}$$

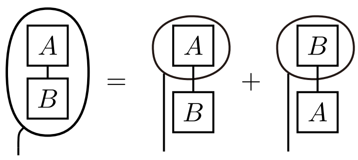


(2.15)

$\updownarrow$

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2\vec{A}$$

The last example to be illustrated is $\nabla(A \cdot B)$, probably the most tricky identity to be both derived and remembered. By the product law, the differentiation is distributed one by one. Here the author utilized the notation for gradient of a vector field (refer to Eq. 2.11).

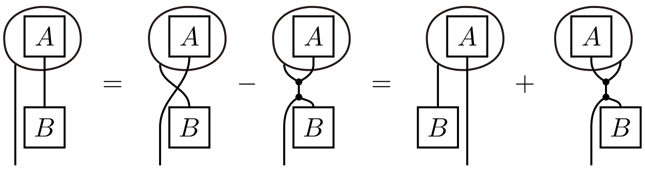


(2.16)

$\updownarrow$

$$\nabla(\vec{A} \cdot \vec{B}) = (\nabla\vec{A}) \cdot \vec{B} + (\nabla\vec{B}) \cdot \vec{A}$$

The next step is a bit tricky, because the contracted epsilon identity is applied reversely, from one pair of the Kronecker deltas to another pair of the Kronecker deltas and one contracted epsilons. In most cases, the identity is applied to break down one contracted epsilons into two pairs of the Kronecker deltas. In this case, however, the original expression is somewhat complicated and it can be recognized that switching them, namely $||\rightarrow$ X, would result in a simpler expression. Let us call this idea as the "line-unweaving" approach. By applying the contracted epsilon identity, $|| =$ X $-$ Y,



(2.17)

$$= \vec{B} \cdot \nabla\vec{A} + \vec{B} \times (\nabla \times \vec{A})$$

In the second equality, two lines of the upper epsilon is swapped and the term is negated. Finally, we get the identity

$$\nabla\big(\vec{A} \cdot \vec{B}\big) = (\vec{B} \cdot \nabla)\vec{A} + \vec{B} \times (\nabla \times \vec{A}) + (\vec{A} \leftrightarrow \vec{B}), \tag{2.18}$$

where “$+(\vec{A} \leftrightarrow \vec{B})$” means adding the same expression with $\vec{A}$ and $\vec{B}$ interchanged.

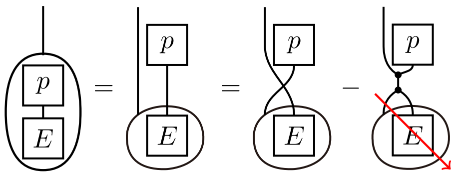
The “line-unweaving” approach is often used, not limited to $||=\times-\asymp$. More examples utilizing the same approach would be introduced in following chapters (Ch. 3.1, 4.3, 4.4). As shown above, in the graphical notation, it is quite clear that unweaving the lines would result in a more graspable expression, whereas in the plaintext notation it is hard to find the motivation of the trick.

Chapter 3. Applications

Here two applications of the graphical notation are presented concisely. They are taken from the author's previous publication.[13] For more such applications, please consult with [13, 14].

3.1 Force Acting On A Point Dipole

Not only for deriving purely mathematical identities, graphical notation is directly applicable to solving problems in physics. The first example is the electrostatic force acting on a dipole. The potential energy U of a point dipole in the electric field $\vec{E}(\vec{r})$ is given by $U = -\vec{p} \cdot \vec{E}(\vec{r})$ and hence the force is given by $\vec{F} = -\nabla U = \nabla(\vec{p} \cdot \vec{E}(\vec{r}))$. To compute this, because $\vec{p}$ is not a function of spatial coordinates, applying the identity for $\nabla(\vec{A} \cdot \vec{B})$ is somewhat unnecessary and lengthy. Applying differentiation to $\vec{E}$ gives what needs the line-unweaving approach.

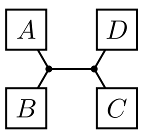


(3.1)

The second term becomes 0 from one of the Maxwell equations, $\nabla \times \vec{E} = 0$, from the assumption of electrostatics. Note that it is evident "at a glance to the practiced eye" in the graphical notation, whereas it normally needs some time to be recognized in the corresponding plaintext notation, $\partial_i(E_j)\, p_l\, \varepsilon_{ijk}\, \varepsilon_{klm}$.

3.2 Different Readings Of The Same Graph

As shown in the embodiment of the cyclic symmetry of the scalar triple product as the rotational symmetry of the graph, graphical notation is a powerful tool to clarify the internal structure of tensorial expressions. Another simple but clear illustration of such advantage is the case when a graph is read with different ways.



$$= \left(\vec{A} \times \vec{B}\right) \cdot \left(\vec{C} \times \vec{D}\right) = \vec{B} \cdot \left(\left(\vec{C} \times \vec{D}\right) \times \vec{A}\right) = \left(\vec{D} \times \left(\vec{A} \times \vec{B}\right)\right) \cdot \vec{C} \qquad (3.2)$$

When $(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D})$ is notated graphically, it is obvious that it can be read either $\vec{B} \cdot ((\vec{C} \times \vec{D}) \times \vec{A})$ or $(\vec{D} \times (\vec{A} \times \vec{B})) \cdot \vec{C}$. When notated with the plaintext notation, however, it is not evident at first glace and needs a little bit of effort to be justified. Such identities sometimes appear in the middle of solving physics problems as well, and the graphical notation can help students not be bothered by mere index-changing processes.

Chapter 4. Generalization

So far, the graphical notation has been applied to only three-dimensional Euclidean vector space. There are many directions of generalization. As mentioned in the introduction, there are other graphical notations which deal with tensor calculus[18, 19, 20, 21, 23], group theory[28, 7, 9], linear algebra[24, 25, 27], quantum mechanics[3, 5], etc. Here the author would like to review some of such graphical notations by adding more notational manipulations on the notation that has been introduced through this thesis. Even though it is mostly review of other notations introduced in previous publications, the author conceive that this approach has a pedagogical value that connects the graphical notation for three dimensional vector calculus and more generalized ones by adding notational rules, showing the relation of different notations and how a specific graphical notation can be built from scratch.

4.1 Index Hierarchy and The Metric

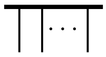
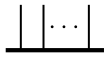
In Euclidean vector space, it is unnecessary to introduce index hierarchy, or the differentiation between upper and lower indices, because a vector and its 1-form share the same numerical components by the definition of the length in the space. In Minkowski spacetime, however, the definition of the spacetime interval is given differently, $(\Delta s)^2 := -(\Delta ct)^2 + \sum_{i=1}^{3}(\Delta x^i)^2$ (or its negative depending on which convention to follow). Defining the metric $g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ gives a concise expression $(\Delta s)^2 = g_{\mu\nu}\Delta x^\mu \Delta x^\nu = \Delta x_\mu \Delta x^\mu$ where $x_\mu = g_{\mu\nu}x^\nu$ and $x^\mu = g^{\mu\nu}x_\nu$ ($g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$). An upper-index is called to be contravariant, and a lower-index covariant. In this way, the differentiation between upper and lower indices, or the index hierarchy, is introduced.

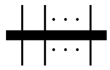
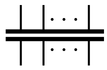
In Penrose's notational convention, the index hierarchy is implemented through the vertical asymmetry of graphical space, similarly to that of plaintext notation. An upper index is denoted as a heading-up line (called an "arm" by Penrose) and a lower index a heading-down line (called a "leg"). The design of the metric $g_{\mu\nu}$ and $g^{\mu\nu}$ is simply a cap($\cap$) and a cup($\cup$), respectively. They embody the nature of the metric well; when $\cap$ is conjoined with a heading-up line (an upper index), the line becomes heading-down (a lower index), and similarly $\cup$ raises a lower index into an upper index. Note that the Kronecker delta is now a straight vertical line.

$$g_{\mu\nu} = \bigcap_{\mu\ \ \nu}, \quad g^{\mu\nu} = \bigcup^{\mu\ \ \nu}, \quad g_{\mu\nu}\, g^{\nu\lambda} = \bigcap_{\mu}\bigcup^{\lambda} = \Big|^{\lambda}_{\mu} = \delta^\lambda_\mu, \quad g_{\mu\nu}\, g^{\nu\mu} = \bigcirc = \delta^\mu_\mu = 4 \tag{4.1}$$

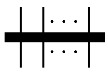
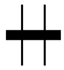
4.2 The General Levi-Civita and the Antisymmetrization

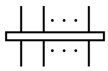
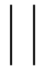
Now that we have a vertical hierarchy in the graph space, notations for more operations should be introduced as well. Again, these notations stick to Penrose's. First, the general n-dimensional Levi-Civita symbol is denoted as a thick horizontal line with n lines, either heading-up or heading-down depending on its contra/co-variance. The indices are read from left to right. The advantage of using this shape for the general Levi-Civita symbol comes from the conciseness and consistency of its the notational relation with the index antisymmetrization. The n-index antisymmetrization is graphically denoted as n lines struck through by a thick black line. The relation between them are shown below. Here we stick to Penrose's notational convention as well. For the case of index antisymmetrization, it is $n!$ times the usual plaintext definition for simplifying normalization issue.

$\varepsilon_{i_1,\cdots,i_n} =$  , $\varepsilon^{i_1,\cdots,i_n} =$  (4.2)

$n!\ \delta^{i_1}{}_{[j_1}\delta^{i_2}{}_{j2}\cdots\delta^{i_n}{}_{j_n]} =$  $=$  $= \varepsilon_{j_1,\cdots,j_n}\,\varepsilon^{i_1,\cdots,i_n}$ (4.3)

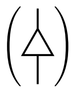
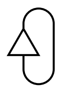
Meanwhile, for the case of symmetrization, thick white lines are used for making it easy to be drawn in computers, but one can always stick to a drawing-friendly convention (a thick wiggly line, for example) depending on their preference. Here the simplest two-index (anti-)symmetrization relations are explicitly shown.

$n!\ \delta^{i_1}{}_{[j_1}\delta^{i_2}{}_{j2}\cdots\delta^{i_n}{}_{j_n]} =$  , $\delta^a{}_b\delta^c{}_d - \delta^a{}_d\delta^c{}_b =$  $=$  $-$  (4.4)

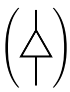
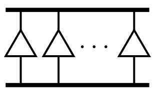
$n!\ \delta^{i_1}{}_{(j_1}\delta^{i_2}{}_{j2}\cdots\delta^{i_n}{}_{j_n)} =$  , $\delta^a{}_b\delta^c{}_d + \delta^a{}_d\delta^c{}_b =$  $=$  $+$  (4.5)

Following the notation just introduced, some important linear algebraic quantities can be expressed and calculated graphically. The trace, the determinant and the inverse of a n by n linear transformation can be expressed graphically. A vertically asymmetric shape is used to denote the matrix to prevent confusion.

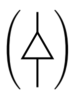
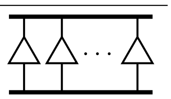
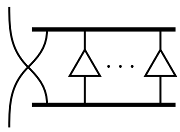
- Trace of an n by n matrix: $\mathrm{Tr}(A^i{}_j) = A^i{}_i$.

Tr  $=$  (4.6)

- Determinant of an n by n matrix: $\det(A) = \varepsilon^{i_1,\cdots,i_n}\ A^1{}_{i_1}\cdots A^n{}_{i_n}$; equivalently, $\det(A) = \frac{1}{n!}\varepsilon^{i_1,\cdots,i_n}\varepsilon_{j_1,\cdots,j_n}A^{j_1}{}_{i_1}\cdots A^{j_n}{}_{i_n}$.

det  $= \dfrac{1}{n!}$  (4.7)

- Inverse of a n by n matrix:
$(A^{-1})^i{}_j = \frac{1}{\det(A)}\ \mathrm{cof}(A)^i{}_j = \frac{1}{\det(A)}\ \frac{1}{(n-1)!}\ \sum_{i_k,j_k=1}^{n-1} \varepsilon^{i,\ i_1,\cdots,i_{n-1}}\varepsilon_{j,\ j_1,\cdots,j_{n-1}}\ A^{j_1}{}_{i_1}\cdots A^{j_{n-1}}{}_{i_{n-1}}$

 $^{-1} = \dfrac{n}{}$   (4.8)

For more detailed application and extension, please consult with Ch. 14 of [21] and [24, 25, 26].

4.3 The Dirac Gamma Matrices

The Dirac equation is concisely notated as $(i\gamma^\mu \partial_\mu - m)\psi = 0$, where γ^μ is a set of four by four matrices satisfying the anticommutation relation $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} 1_{4\times4}$ where $1_{4\times4}$ is a four by four identity matrix, diag(1, 1, 1, 1). In Dirac representation, gamma matrices are defined as follows:

$$\gamma^0 = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & +1 \\ 0 & 0 & +1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

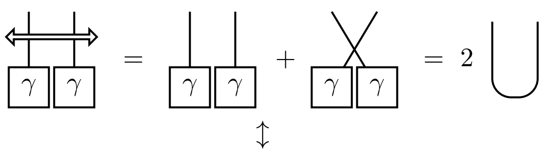
$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & +i & 0 \\ 0 & +i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \end{pmatrix}$$

The anticommutation relation is the fundamental property defining gamma matrices. Many properties of them can be derived from the anticommutation relation, without referring to exact numbers of the matrix components. It is, however, not really easy to calculate explicitly with plaintext notation mainly because of bulkiness and intractability of indices. In graphical notation, however, it is relatively easy and tractable. The graphical notation and derivation of gamma matrices' properties have been introduced by Kennedy and Cvitanović.[8, 11, 12] In this section, the most basic calculations of such works are presented.

Before getting started, it should be reminded that one gamma matrix is itself a four by four matrix, which carries two index lines for matrix components. On top of that, there is another index to enumerate gamma matrices. In that sense, gamma matrices should have three index lines! For simplicity, however, we will omit two index lines for matrix components but rather add a horizontal asymmetry which distinguishes "left" and "right", so that matrices should be fixed horizontally and interpreted in the order (conventionally, lines are used to denote matrix index lines). Similarly, the identity matrix $1_{4\times4}$ will be omitted without loss of generality. It can be put again if needed, especially when taking the trace: $\mathrm{Tr}(1_{4\times4}) = \mathrm{Tr}(\delta^\mu_\nu) = \delta^\mu_\mu = 4$.

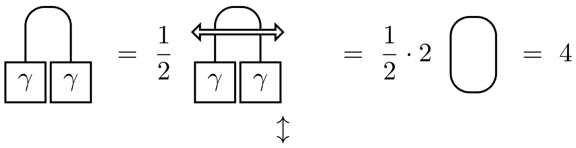
The notation of symmetrization with a small change—arrow heads— is to be used as the notation of anticommutation here. For real, if we deal only with the same kind of objects, they can be the same; $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\ \gamma^{(\mu}\gamma^{\nu)}$. In general, however, anticommutation should be differentiated from symmetrization, because $\{\alpha^\mu, \beta^\nu\} = \alpha^\mu\beta^\nu + \beta^\nu\alpha^\mu \neq \alpha^\mu\beta^\nu + \alpha^\nu\beta^\mu = 2\ \alpha^{(\mu}\beta^{\nu)}$, if α and β do not commute.

Starting from the fundamental anticommutation relation of gamma matrices, $\gamma^\mu\gamma_\mu = 4$ can be derived effortlessly.



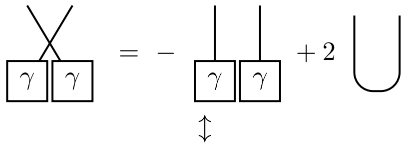
(4.9)

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\ g^{\mu\nu}$$



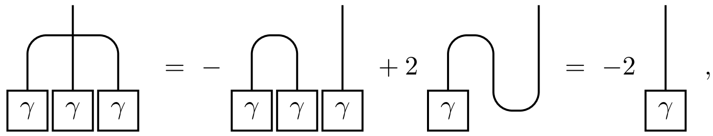
$$\gamma^\mu \gamma_\mu = \frac{1}{2} g_{\mu\nu}\{\gamma^\mu, \gamma^\nu\} = \frac{1}{2} \cdot 2\, \delta^\mu{}_\mu = 4 \tag{4.10}$$

To derive some other identities, the "line-unweaving" approach is needed. The unweaving rule can be taken from Eq. 4.11 as follows:

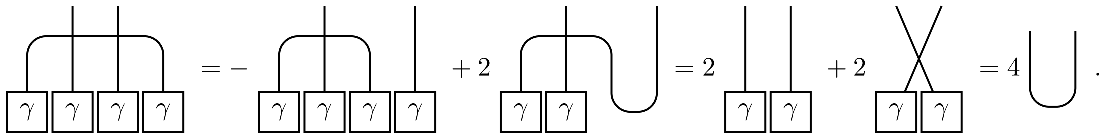


$$\gamma^\nu \gamma^\mu = -\gamma^\mu \gamma^\nu + 2\, g^{\mu\nu}. \tag{4.11}$$

Utilizing the unweaving rule, following identities can be proven easily: $\gamma_\mu \gamma^\nu \gamma^\mu = -2\gamma^\nu$, $\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\mu = 4g^{\nu\lambda}$, $\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\sigma \gamma^\mu = -2\gamma^\sigma \gamma^\lambda \gamma^\nu$. The first and the second identities' proofs are as follows:



(4.12)



(4.13)

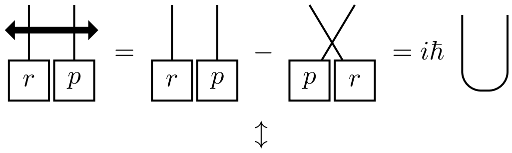
Eq. 4.10 and Eq. 4.11 are utilized for the last part of Eq. 4.12 and Eq. 4.13, respectively.

More identities regarding gamma matrices, such as traces of the matrices or the matrices contracted with four vectors are readily solvable by just connecting (here omitted) matrix indices and attaching vectors to gamma matrice's index line, respectively. This graphical notation and approach enable the equations to be understood promptly and clearly, by showing how the calculation process goes visually without referring to bulky indices.

4.4 Commutation Relations in Quantum Mechanics

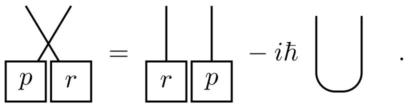
So far in Ch. 4, the index hierarchy is kept turned on to differentiate contra/co-variance. In this section, however, let the index hierarchy turned off but keep the horizontal asymmetry turned on, because quantum mechanical operators are generally not commutable.

The canonical commutation relation, $[\hat{r}_i, \hat{p}_j] = i\hbar\, \delta_{ij}$ is the fundamental commutation relation and used to derive the angular momentum operator's commutation relation. Even though the algebraic steps are elementary, going through the process in plaintext notation can be tedious and one might not find the trick promptly. In graphical notation, the whole process is fairly evident. Starting from the canonical commutation relation,



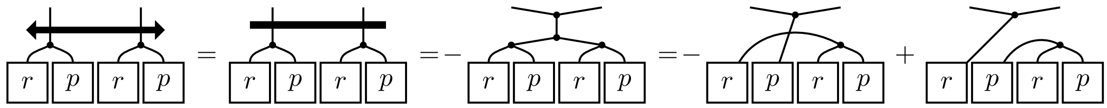
$$[\hat{r}_i,\, \hat{p}_j] \;=\; \hat{r}_i\, \hat{p}_j \;-\; \hat{p}_j\, \hat{r}_i \;=\; i\hbar\, \delta_{ij} \quad , \tag{4.14}$$

the following unweaving rule is confirmed:

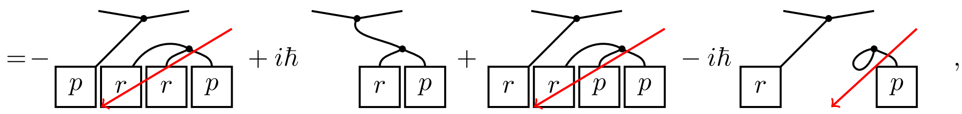


(4.15)

Utilizing the unweaving rule between $\hat{r}$ and $\hat{p}$, the commutation of angular momentum operations, $[\hat{L}_i,\ \hat{L}_j] = [(\hat{r}\times\hat{p})_i,\ (\hat{r}\times\hat{p})_j]$, can be calculated as:



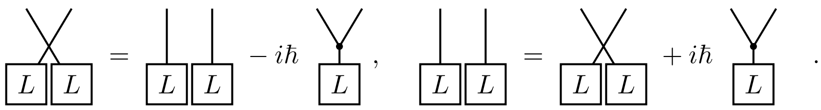
(4.16)



(4.17)

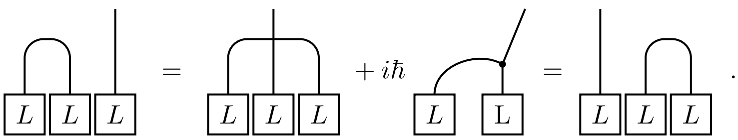
so that $[\hat{L}_i,\ \hat{L}_j] \;=\; i\hbar\, \varepsilon_{ijk}\, \hat{L}_k$. Note that, in Eq. 4.16, the motivation to attack one of the contracted epsilons is obvious and the process is clear. In Eq. 4.17, terms except the 2nd one is zero because of having symmetric and antisymmetric properties at the same time (refer to Eq. 2.12).

Now that we have a commutation relation between angular momentum operators, we can similarly obtain the commutation relation between the angular momentum operator and the square of the total angular momentum. From the commutation relation $[\hat{L}_i,\ \hat{L}_j] \;=\; i\hbar\, \varepsilon_{ijk}\, \hat{L}_k$, the unweaving rule is taken, together with the weaving rule this time:



(4.18)

Applying the rules consecutively to $\hat{L}^2\, \hat{L}_i$ gives the commutation relation $[\hat{L}^2,\ \hat{L}_i] = 0$, or equivalently $\hat{L}^2\, \hat{L}_i = \hat{L}_i\, \hat{L}^2$, as follows:



(4.19)

The first equality comes from the weaving rule, and the second from the unweaving rule. The strength of graphical notation here is that it effectively carries only the essential information so that both calculation and its understanding can be clearer and faster than traditional plaintext notation, especially when dealing with more terms and indices.

Chapter 5. Conclusion

So far, the efficiency and applicability of the graphical notation is illustrated by providing graphical rules and proofs of vector calculus identities in three-dimensional Euclidean space. Its modification for more generalized use is also discussed in detail in the context of tensor/linear algebra, gamma matrices' relation and the commutation relation in quantum mechanics, so that readers are encouraged to use and devise graphical notations by themselves based on their need.

Graphical notation brings many advantages. First, it visualizes equations and consequently makes it more memorable and intuitively manipulable. The speed of calculation is boosted as well. Second, the desirable direction of calculation is often more evident in graphical notation. One can get a hint from the connectivity of lines, not from reading and matching every single index. Third, it is readily translatable to either vectorial or plaintext notation. In principle, graphical notation can be used as the standard language for vectorial/tensorial calculation, because of its readability and efficiency as well as easy translation. Fourth, the concept of tensors is naturally introduced, where its calculation and manipulation is still clear in graphical notation. Lastly, it can function as the exemplified "toy model" of graphical notation, so that students can adapt themselves into graphical notation and study more advanced topics by modifying and developing what they are already familiar with.

Historically speaking, Euclidean vector calculus has not been the topic of graphical notation, of which the reason is not certain. Some plausible explanations might be (1) there have been few researchers who actively use graphical notation in general, and "mere" vector calculus have not been their main interest because it is already simple enough or has not many topics to conduct research; (2) the advantage of and the potential demand on graphical notation in vector calculus is underestimated so that there has been low motivation to introduce it.

It does not seem probable that no one used graphical notation for vector calculus before the work of Kim, Oh and Kim [13]. Rather, it seems to be a coincidence that previous publications did not cover vector calculus in graphical notation. In that sense, this thesis and the material provided by Kim et al. [13, 14] would hold a high pedagogical and educational value because undergraduate level vector calculus itself is concisely and efficiently introduced in the graphical language, as well as its further applications and generalizatoins are provided for more advanced topics. To that end, the author hopes for many educators and students to find these materials useful and enjoyable. Graphical notation would change the activities of vector calculus classes from index-reading and formula-memorizing to graph-drawing and prompt formula-deriving, which would be more creative and interesting.

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Summary

Graphical Notation for Vector Calculus and Its Generalization

그래프적 표기법은 물리학에서 오랜 기간 동안 소개되고 사용되어왔다. 그러나 그 효율성, 가독성, 응용가능성에도 불구하고 여전히 전통적인 줄글 표기법에 비해 소수로 남아있다. 이 학위논문에서는 벡터 미적분학을 위한 그래프적 표기법을 리뷰하고 일반화하였다.

벡터 미적분학의 예시를 통해 그래프적 표기법의 본질과 장점을 보였다. 근본적인 규칙을 소개하고 응용하여 대표적인 벡터 미적분학 항등식을 증명하였으며, 이를 통해 그래프적 표기법이 얼마나 간결하고 효율적인지 실증하였다. 감마 매트릭스 연산과 양자역학적 교환관계 등과 같은 더 일반화된 개념과 표기법을 리뷰함으로써 그래프적 표기법이 어떻게 변경되고 개인의 필요에 맞춰질 수 있는지 보였다.

그래프적 표기법이 학부 수준 벡터 미적분학에서 손쉽게 적용되고 가르쳐질 수 있도록 만듦으로써, 그래프적 표기법의 진입 장벽이 낮아지고 실질적인 사용이 촉진될 것이다. 일반화 예시들은 독자들이 스스로 그래프적 표기법을 조작하고 개발할 수 있도록 장려할 것이며, 이는 그래프적 표기법의 사용을 확대하고 다양화할 것이다. 이는 물리적 개념과 실체가 더 구체적으로 이해되고 묘사되도록 도울 것이다.

감 사 의 글

연구뿐만 아니라 진로와 생활 등에도 항상 지도와 조언을 아끼지 않으신 김근영 지도교수님께 감사인사를 전합니다. 연구에 관해 지도를 부탁드렸을 때 긍정적으로 독려해주시고, 이후로도 잘 지도해주셔서 연구도 대학생활도 잘 마무리할 수 있게 되었습니다.

관련 연구를 함께 수행하고 논문 출판 작업을 하며 항상 지적으로 많은 귀감이 된 친구 김준휘에게도 고마움을 전합니다. 덕분에 스스로의 미숙함을 보완하면서 더 넓게 이해하고 공부할 수 있는 계기가 되었습니다.

학부 재학 중에 수행했던 다른 연구들도 제가 경험과 실력을 쌓고 지적으로 성장하는 데에 많은 도움이 되었습니다. 학부 과정 동안 흥미로운 연구를 할 수 있도록 지도하고 배려해주셨던 분들께 감사인사를 전합니다. 노도영 교수님(GIST, 2017-2018), Professor Philip Hopkins and Doctor Coral Wheeler (Caltech, 2019), Professor Yannis Semertzidis and TA Merve Yıldız (KAIST, 2019), 이광록 교수님 (GIST, 2017), thank you for all of your care and guidance!

멋진 학습 환경과 기회—특히 연구친화적 문화와 여러 글로벌 프로그램—를 만들고 제공해주신 지스트 대학과 교직원 분들께도 감사를 전합니다. 덕분에 다른 곳에선 누릴 수 없을만한 혜택을 누렸습니다. 평생의 마음의 짐으로 삼고, 그것을 다시 사회에 보답하고 환원하는 삶을 살도록 노력하겠습니다.

대학 생활동안 사귄 친구들, 특히 기숙사 3층 회의실에서 동고동락한 "삼회늘그니들" 친구들에게도 고마움을 표합니다. 대학 생활동안 마음 맞는 친구들과 함께 재미있고 의미있는 시간을 보낼 수 있어서 참 좋았습니다.

항상 저를 믿고 지지해주시며 지켜봐주시는 든든한 가족에게도 감사의 말을 전합니다. 표현이 서툴러 자주 표현하지 못하지만, 제가 가고자 하는 길을 응원해주고 기쁨도 슬픔도 함께 나눌 가족이 있기에 대학생활을 하며 자유롭게 공부하고 계속해서 꿈을 추구해나갈 수 있었습니다.

마지막으로, 제가 스스로의 태도와 가치관에 대한 의심과 불확실함을 거둘 수 있도록 시간과 공간의 제약을 넘어 많은 영향을 준 Richard Feynman에게도 감사를 표하고 싶습니다. Without you, I might have lost myself and my identity, having no lighthouse in my mind. Here I promise that I will do my best to become a maverick lighthouse to others, as you are to me.

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연 구 업 적

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